PRELIMINARY TO PCF

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Abstract. Basic notions for the course on PCF: stationary sets, clubs, club guessing sequences etc. Proofs were taken from the book Set Theory by Jech, Handbook of Set Theory, chapter on cardinal arithmetic by Abraham-Magidor and from Non-Existence of Universal Members (Sh820) by Shelah.

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1. Cofinality and König's Theorem

Definition 1. 1.For limit ordinals α, β we say that the increasing sequence $\langle \alpha_\gamma : \gamma < \beta \rangle$ is cofinal in α if $\lim\limits_{\gamma \to \beta} \alpha_\gamma = \alpha$. We say that the cofinality of α is β, if β is the least limit ordinal such that an increasing β cofinal sequence exists in α.

2.We can also say that $A \subset \alpha$is cofinal in α if $sup(A) = \alpha$.

Claim 2. $cf(cf(\alpha)) = cf(\alpha)$.

Proof. If $cf(\alpha) = \beta$ the there exists a sequence $\langle \alpha_i : i < \beta \rangle$ which is a witness for it. Now suppose that $cf(\beta) = \gamma$ as witnessed by $\langle \beta_\zeta : \zeta < \gamma \rangle$, then $< \alpha_{\beta_\zeta} : \zeta < \gamma >$is cofinal in α (so if we assumed that $cf(\beta) < \beta$ we would have had a contradiction). □

Claim 3. For every κ , $cf(\kappa)$ is a cardinal.

Proof. If $\alpha = cf(\kappa)$ is a limit ordinal and not a cardinal, then there is a one to one, onto map $f : \alpha \to |\alpha|$ and we can use this map to define a $|\alpha|$ sequence, cofinal in κ: suppose we are given$\langle \kappa_i : i < \alpha \rangle$ cofinal in κ, and define by induction a new sequence :$\delta_0 = \kappa_0$, $\delta_{i+1} = max(\delta_i + 1, \kappa_{f(i)} + 1)$, for a limit ordinal we define $\delta_i = max(\underset{j<i}{\cup}\delta_j + 1, \kappa_{f(i)} + 1)$ (note that at a limit stage i, the union is less than κ since we assumed $cf(\kappa) = \alpha > |\alpha| \geq i$). □

Remark 4. The proof actually demonstrates that the definition is the same if we defined $cf(\kappa)$ to be the least cardinal such that there is a cofinal set of such cardinality in κ.

Definition 5. We define a cardinal κ to be regular if $cf(\kappa) = \kappa$.

Exercise 6. Let us draw a short table, to try and understand how the class of cardinals divides. Try to fill the table with examples of cardinals that provide the cross properties and are uncountable. If you can't find one, try to explain why.

	successor	limit
singular		
regular		

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Theorem 7. *(KÖNIG) Assume I is an infinite set, and for every $i \in I$ $\kappa_i < \lambda_i$then:* $\sum_{i\in I}\kappa_i < \prod_{i\in I}\lambda_i$.

Proof. Assume first that we had: $\sum_{i\in I}\kappa_i \geq \prod_{i\in I}\lambda_i$. Then we would have an onto function f from the L.H.S to the R.H.S. In that case we would have: $\prod_{i\in I}\lambda_i = \cup_{i\in I} S_i$ where $S_i = f(\kappa_i)$ and hence $|S_i| \leq \kappa_i < \lambda_i$ ($\kappa_i$here is a set so this is an abuse of notation). So if we could prove that for any such union of sets $\prod_{i\in I}\lambda_i \neq \cup_{i\in I} S_i$ then we are done. Assume then that $\cup_{i\in I} S_i \subset \prod_{i\in I}\lambda_i$ and look at the projection of S_i onto the i'th coordinate namely: $S_i(i) := \{h(i) : h \in S_i\}$. Since $|S_i| < \lambda_i$ we have for every i $S_i(i) \subsetneqq \lambda_i$and choose $h \in \prod_{i\in I}\lambda_i$ such that $h(i) \notin S_i(i)$. So $h \notin \cup_{i\in I} S_i$ and we are done. □

Corollary 8. *For every infinite cardinal κ we have $\kappa^{cf(\kappa)} > \kappa$.*

Proof. Let $\kappa = \sum_{i<cf(\kappa)}\kappa_i$ where $\kappa_i < \kappa$ then $\kappa = \sum_{i<cf(\kappa)}\kappa_i < \prod_{i<cf(\kappa)}\kappa = \kappa^{cf(\kappa)}$. □

Exercise 9. Use König's Theorem to prove that $\kappa < 2^\kappa$

Corollary 10. *For every infinite κ we have $cf(2^\kappa) > \kappa$.*

Proof. We use the fact that $(2^\mu)^\mu = 2^{\mu\times\mu} = 2^\mu$ for any infinite cardinal μ. Assume now that for every $i < \kappa$ we have $a_i < 2^\kappa$, then $\sum_{i<\kappa} a_i < \prod_{i<\kappa} 2^\kappa = (2^\kappa)^\kappa$. □

It may be worth mentioning at this point a

Theorem 11. *(Easton) Let M be a transitive model of ZFC and assume that the Generalized Continuum Hypothesis holds in M (namely that $2^\kappa = \kappa^+$ for every infinite cardinal κ). Let F be a function in M from the regular cardinals to the cardinals such that:*
1. $F(\kappa) > \kappa$.
2. $F(\kappa) \leq F(\lambda)$ *whenever* $\kappa \leq \lambda$.
3. $cf(F(\kappa)) > \kappa$.

Then there is a generic extension of M with the same cardinals and cofinalities, that realizes F as the continuum function on regular cardinals.

2. Closed Unbounded Sets, Stationary Sets.

We start with some definitions:

Definition 12. Let κ be a regular uncountable cardinal. A closed unbounded set (or a club) $C \subset \kappa$ is a subset of κ with the property $sup(C) = \kappa$ and such that C contains all its limit points in κ, namely: that for each limit ordinal $\alpha < \kappa$ we have $(sup(C \cap \alpha) = \alpha) \implies \alpha \in \kappa$.

Exercise 13. If C and D are clubs then so is $C \cap D$. (Notice that this is not true for $\kappa = \aleph_0$).

Remark 14. The fact that the empty set is not a club, that the set of clubs are closed under intersection and that the set κ is a club suggests that we can define:

Definition 15. The Closed Unbounded Filter on κ, is the set of all subsets of κ that contain a club.

Claim 16. The club filter on κ is κ complete.

Proof. We only need to prove that the intersection of less than κ clubs is again a club. So let $\gamma < \kappa$ and for every $\alpha < \gamma$ let C_α be a club. We first note that if we replace C_α with $D_\alpha = \underset{\beta<\alpha}{\cap} C_\beta$ then we still have: $\underset{\alpha<\gamma}{\cap} D_\alpha = \underset{\alpha<\gamma}{\cap} C_\alpha$. We proceed by induction on $\alpha \leq \gamma$ to prove that $\underset{\beta<\alpha}{\cap} D_\beta$ is a club. For $\alpha = 0$ this is given. On successor stage this is the content of the exercise. At a limit ordinal δ we have: $D_0 \supset D_1 \supset D_2 \supset .... \supset D_\delta = \underset{\beta<\delta}{\cap} D_\beta$ a descending series of clubs accept maybe for D_δ. It is an exercise to see that D_δ is closed. We prove that it is unbounded: Let $\eta < \kappa$ and choose a sequence: $\eta < a_0 < a_1 < ...$ inductively such that $a_\beta \in D_\beta$, $a_\beta \geq sup\{a_i : i < \beta\}$, $a_0 > \eta$. This is possible because of the induction hypothesis, and the fact that κ is regular. The resulting sequence is bounded in κ and hence has a limit. For Evey β, $< a_\xi : \beta < \xi < \delta > \subset D_\beta$ and since D_β is closed it contains this limit, and we are done. □

Definition 17. Let κ be an uncountable regular cardinal. We call a set $S \subset \kappa$ stationary if for every club C we have $C \cap S \neq \emptyset$.

Remark 18. On one hand stationary implies that we are dealing with a "big" set. However, Notice that according to Exercise 12, the complement of a club cannot contain a club. The situation is not the same for stationary sets, since for κ regular less then λ, we can define $E^\lambda_\kappa := \{\delta < \lambda : \delta\, is\, a\, limit\, ordinal\, and\, cf(\delta) = \kappa\}$ which is a stationary set (exercise) and if $\lambda > \kappa > \aleph_0$ and κ is regular, then the sets E^λ_κ, $E^\lambda_{\aleph_0}$ are disjoint. But in fact we get much more, as we shall see later.

Definition 19. A function f on a set of ordinals S is regressive, if for every $\beta \in S$ we have $f(\beta) < \beta$.

Theorem 20. *(Fodor) if f is a regressive function which domain is a stationary set S, then there is $S' \subset S$ which is also a stationary set, and on which f is constant.*

We might skip the proof, but we state a result of the Fodor Theorem:

Exercise 21. Every stationary set contains either a stationary set which is also contained in E^λ_κ for some $\kappa < \lambda$, or a stationary set which is also contained in $E^\lambda_{reg} = \{\delta < \lambda : \delta\, is\, a\, limit\, cardinal\, and\, cf(\delta) = \delta\}$ (and notice that the latter set is not necessarily stationary).

Theorem 22. *Let $\lambda > \kappa$ be regular cardinals. Every stationary subset of E^λ_κ is the union of λ disjoint stationary sets.*

Proof. Let $S \subset E^\lambda_\kappa$ be as above. Since every $\alpha \in S$ has cofinality κ, we can choose for every α an increasing sequence $\langle a^\alpha_i : i < \kappa\rangle$ with limit α. Then if for some $i < \kappa$ we had that for every $\beta < \kappa$ the set $W_\beta := \{\alpha \in S : a^\alpha_i \geq \beta\}$ is stationary, we could define a regressive function with domain S such that $f(\alpha) = a^\alpha_i$. The function f is regressive, defined on S, and for each $\beta < \kappa$ we have $W_\beta \subset S$. So for each $\beta < \kappa$ we can find a stationary subset $S_{\gamma_\beta} \subset W_\beta$ such that f is constant with value $\gamma_\beta \geq \beta$. Regularity of λ implies that we obtain λ such different values, and since $\gamma_{\beta_1} \neq \gamma_{\beta_2}$ implies $S_{\gamma_{\beta_1}} \cap S_{\gamma_{\beta_2}} = \emptyset$, we have the required result. It remains to show that indeed we can find such $i < \kappa$ such that for every $\beta < \kappa$ the set $W_\beta := \{\alpha \in S : a_i^\alpha \geq \beta\}$ is stationary. Assume towards contradiction that this is not the case, then for every $i < \kappa$ there are β_i and C_i such that $a_i^\alpha < \beta_i$ for every $\alpha \in C_i \cap S$. Let β be the supremum over all β_i, which is less than λ, and $C = \underset{i<\kappa}{\cap} C_i$, then we have on one hand that C is a club and hence $C \cap S$ is stationary, and on the other hand that for all $i < \kappa$ and $\alpha \in C \cap S$, $a_i^\alpha < \beta$, a contradiction. □

Remark 23. We may want to try and generalize our results to singular cardinals. Let μ be a singular cardinal with $\mu > cf(\mu) = \kappa > \omega$, we wish to define by the same way closed unbounded sets, stationary sets etc. Most of the results go through, accept for Fodor's Theorem. We deal with this in the following definition and theorem:

Definition 24. 1. Let $\mu > cf(\mu) = \kappa > \omega$. By a continuous unbounded embedding of κ into μ we mean a one to one, increasing function $e : \kappa \to \mu$ such that $Im(\kappa)$ is cofinal in μ (unbounded) and such that for every limit ordinal $\delta < \kappa$ we have: $e(\delta) = \underset{\beta<\delta}{\cup} e(\beta)$ (continuous). In that case we note that the image of e is a club of μ, and hence if S is stationary in μ then $S \cap Im(e)$ is again stationary, and that its preimage under e is stationary in κ.

Theorem 25. *(Fodor's Theorem For Singular Cardinals) Assume that $\mu > cf(\mu) = \kappa > \omega$ and $S \subset \mu$ is stationary, and that $f : S \to \mu$ is regressive. Then there exists an $S' \subset S$ which is also stationary and on which f is bounded.*

Proof. Without loss of generality $S \subset Im(e)$. Now look at $S' \subset S$ which is again stationary. Now denote $T = e^{-1}(S')$ which is stationary in κ and define: $g : T \to \kappa$ by $g(\alpha) = \underset{\beta<\kappa}{Min}(e(\beta) \geq f(e(\alpha))$. Since f is regressive, and since α is a limit ordinal (remember, $\alpha \in T$ $e(\alpha) \in S'$) we have that $f(e(\alpha)) < e(\alpha) \to e(\beta) < e(\alpha)$ which means that g is regressive. Now apply the Fodor Theorem to get the necessary result. □

Definition 26. For $S \subset \kappa$ stationary, a club guessing sequence $\langle C_\delta : \delta \in S\rangle$is such that each $C_\delta \subset \delta$ is closed unbounded in δ, and such that for each club $D \subset \kappa$ there exists a δ such that $C_\delta \subset D$.

Theorem 27. *For a regular cardinal κ and a cardinal λ with $cf(\lambda) \geq \kappa^{++}$ any stationary $S \subset E_\kappa^\lambda$ has a club guessing sequence $\langle C_\delta : \delta \in S\rangle$and C_δ is of order type κ.*

Proof. The proof given here is for an uncountable κ only.

Let S be as above, and fix any sequence $C = \langle C_\delta : \delta \in S\rangle$with $C_\delta \subset \delta$, $otp(C_\delta) = \kappa$ closed and unbounded in δ. For any D which is a club in λ define the (double) reduction: $C \,|\, D := \langle C_\delta \cap D : \delta \in S \cap D'\rangle$ where $D' = Acc(D) \cap D = Acc(D)$. We intersect with D' to ensure that $C_\delta$is closed and unbounded in δ. We claim that for some club of λ, this produces a club guessing sequence (once such a partial (to S) sequence is obtained, we can expand it to be defined on S by defining $C_\alpha$to be some cofinal set in α for those elements that were thrown away). Assume that this is not the case, then for every club $D \subset \lambda$ there exist witnesses to the failure of $C \,|\, D$ i.e.: there is another club $E(D)$ such that for every $\delta \in S \cap D'$ we have: $C_\delta \cap D \nsubseteq E(D)$. Define a decreasing sequence of clubs by induction on $\alpha \leq \kappa^+$ as follows:

1. $E^0 = \lambda$

2. For a limit ordinal: $E^\gamma = \cap\{E^\beta : \beta < \gamma\}$ (and refer to a previous claim to see that this is again a club).

3. For a successor ordinal, we define $E^{\beta+1} = (E^\beta \cap E(E^\beta))'$ (the set of accumulation points of the intersection of the previous club with its counter example)

We denote: $E = E^{\kappa^+}$. We now get a contradiction: take any $\delta \in S \cap E$ and look at $C_\delta \cap E$, then since C_δ has cardinality κ and since the sequence $\langle E^\alpha : \alpha < \kappa^+\rangle$ is decreasing and is κ^+ long, there exists an $\alpha < \kappa^+$ with $E^\alpha \cap C_\delta = E \cap C_\delta$. For Every $\beta > \alpha$ this is also true and in particular for $\beta = \alpha + 1$. But since $\delta \in E^{\alpha+1}$ we have $C_\delta \cap E^\alpha \nsubseteq E^{\alpha+1}$. □

At this stage we could prove for instance a

Theorem 28. *Shelah (Sh820): Assume* $\mu^+ < \lambda = cf(\lambda) < \mu^{\aleph_0}$ *Then the class* K^{tr}_λ *has no universal member where:*

1. K^{tr}_λ *is the class of trees of cardinality* λ *with* $\omega + 1$ *levels.*

2. The embeddings are one to one functions such that: I. $t < s \iff f(t) < f(s)$ *and II. lev(x)=lev(f(x)).*

Another result which uses club guessing sequences is given in Universal Abelian Groups, by Kojman-Shelah (no. 455 in the Shelah archive) which states a

Theorem 29. *For* $n \geq 2$ *there is a purely universal separable p-group in* $\aleph_n$ *if and only if* $2^{\aleph_0} < \aleph_n$*.*