Six Properties of Character Varieties

Adam S. Sikora

July 22, 2010

Let G be either •compact, or
•complex, reductive, eg. $SL(n,\mathbb{C})$,...

For a fin. gen. group Γ, let $X_G(\Gamma)$ be the G-character variety of Γ.

For G compact, $X_G(\Gamma) \stackrel{def}{=} Hom(\Gamma, G)/G$.
For complex reductive G,

$$X_G(\Gamma) \stackrel{def}{=} Hom(\Gamma, G)//G.$$

(This is an algebraic set.)

Character varieties are "classical objects" underlying many Quantum Field Theories, e.g. Chern-Simons QFT of Witten. This theory gives quantum invariants of 3-mflds.

1. Algebraic description of $X_G(\Gamma)$.

For algebraic G, $X_G(\Gamma)$ is an algebraic set which can be described by:
(a) an embedding $X_G(\Gamma) \to \mathbb{C}^n$
It requires a finite set of generators of $\mathbb{C}[X_G(\Gamma)]$.
(b) a finite set of polynomial equations on $\mathbb{C}^n$ describing $X_G(\Gamma)$.
This can be done by the Gröbner basis method.

Candidates for generators:
For each $\gamma \in \Gamma$ and $\phi : G \to GL(n, \mathbb{C})$ we have $\tau_{\gamma,\phi} : X_G(\Gamma) \to \mathbb{C}$ sending $\rho : \Gamma \to G$ to $tr\phi\rho(\gamma) \in \mathbb{C}$.

For classical groups, one often considers the defining rep., eg. $SL(n, \mathbb{C}), SO(n, \mathbb{C}) \subset GL(n, \mathbb{C})$ and $\tau_\gamma = \tau_{\gamma,\phi}$.

τ_γ depends on the conjugacy class of γ only. These are the "Wilson loops".

Generators of Character Varieties

$\mathbb{C}[X_G(\Gamma)]$ is generated by τ_γ's for $G = GL(2,\mathbb{C})$, $SL(2,\mathbb{C})$.

Thm $\mathbb{C}[X_G(\Gamma)]$ is generated by τ_γ for all γ for $G = GL(n,\mathbb{C}), SL(n,\mathbb{C}), Sp(n,\mathbb{C}), O(n,\mathbb{C})$, $SO(2n+1,\mathbb{C})$.

Thm (follows from work of Aslaksen-Tan-Zhu) For $n \geq 2$, $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ is generated by τ_γ for all γ and by functions $\lambda_{\gamma_1,...,\gamma_n}$, for $\gamma_1, ..., \gamma_n \in \Gamma$ where λ is obtained by polarization of a function $\Lambda : M(2n,\mathbb{C}) \to \mathbb{C}$, $\Lambda(A) = Pf(A - A^T)$.

Thm(S.) For $n \geq 2$, $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ is not generated by $\tau_{\gamma,\phi}$ for all $\gamma \in \Gamma$ and and all representations ϕ of $SO(2n,\mathbb{C})$.

Reminder: $\tau_{\gamma,\phi}$ sends $\rho : \Gamma \to G$ to $tr\phi\rho(\gamma) \in \mathbb{C}$.

Finite Generating Sets of $\mathbb{C}[X_G(\Gamma)]$.

Thm For $G = SL(n,\mathbb{C}), GL(n,\mathbb{C})$, $\mathbb{C}[X_G(\Gamma)]$ is generated by τ_γ for all γ which are words in at most n^2 generators of Γ.

Conj Words in at most $\binom{n+1}{2}$ generators are enough.

Thm.(S.) Similar finite generating sets of for all classical groups G.

Open: Exceptional groups.

2. Character varieties of the torus.

If G is compact then $X_G(\mathbb{Z}^2) = (T \times T)/W$, for $T =$ max torus in G, W=Weyl group.

Thm (S.) $X_G(torus) = (T \times T)/W$,
for $G = GL(n,\mathbb{C}), SL(n,\mathbb{C}), Sp(n,\mathbb{C}), O(n,\mathbb{C})$,
$SO(2n+1,\mathbb{C})$ but not $SO(2n,\mathbb{C})$.

$X_{SO(2n,\mathbb{C})}(\mathbb{Z}^2)$ is not a normal algebraic set.
Its normalization is $(T \times T)/W$.

3. Tangent Spaces to Character Varieties

Thm $T_{[\rho]}\, X_G(\Gamma) = H^1(\Gamma, \mathfrak{g})$ (twisted by $\Gamma \xrightarrow{Ad} G \xrightarrow{\rho} Gl(\mathfrak{g})$) if two conditions hold.

•$\rho(\Gamma) \subset G$ has the trivial stabilizer.
Irreducibility implies trivial stabilizer for $SL(n, \mathbb{C})$, $SU(n)$, but not other groups.

•ρ is a reduced point of the scheme $\mathcal{X}_G(\Gamma)$ (no nilpotents).
There are groups Γ for which
$dim\, T_{[\rho]}\, X_G(\Gamma) < dim\, H^1(\Gamma, \mathfrak{g})$
for a Zariski open set of ρ's.
In fact, there are 3-mfld groups like this.

Thm(S.) If $T_{[\rho]}\, X_G(\Gamma) = H^1(\Gamma, \mathfrak{g})$ for a Zariski open set of ρ's iff $\mathcal{X}_G(\Gamma)$ is reduced.

Thm(S.) All G-representations of surface groups for all G are reduced.

4.

If Y is a top. space then $X_G(Y) \stackrel{def}{=} X_G(\pi_1(Y))$.

Thm(Goldman) Let F be a closed surface.
If G is compact then $X_G(F)$ is symplectic.
If G is complex, reductive, then $X_G(F)$ is complex symplectic.

Thm If $\partial M^3 = F$ then the image of $r^* X_G(M) \to X_G(F)$ is an isotropic submanifold (with singularities).

For "simple" examples (eg. $M =$handlebody), it is Lagrangian. However, there are reasons to believe that $r^*(X_G(M)) \subset X_G(F)$ is not Lagrangian.

5. A Generalization of Tangent Space Formula

What if the stabilizer S_ρ of $\rho(\Gamma)$ in G is non-trivial?

Thm(S.) If ρ is completely reducible and scheme smooth then

$$T_{[\rho]}\, X_G(\Gamma) \simeq T_0 \left(H^1(\Gamma, Ad\,\rho)//S_\rho \right).$$

Def ρ is "scheme smooth" if it is a smooth point of the algebraic scheme $Hom(\Gamma, G)$.

Heusener and Porti: $G = PSL(2, \mathbb{C})$.

6. Complete Integrability

From physics point of view, character varieties are phase spaces. In this context it is an important question whether they are completely integrable.

Def. For a symplectic mfld, (M, ω), a completely integrable system is $\{f_1, ..., f_n : M \to \mathbb{R}\}$ s.t. $f_1, ..., f_n$ are alg. independent and $\{f_i, f_j\} = 0$ for all i, j, and $n = \frac{1}{2} dim\, M$.

For a complex symplectic mfld, (M, ω), a completely integrable system is $\{f_1, ..., f_n : M \to \mathbb{C}\}$ s.t. $f_1, ..., f_n$ are alg. independent and $\{f_i, f_j\} = 0$ for all i, j, and $n = \frac{1}{2} dim_{\mathbb{C}}\, M$.

Let F be a surface F of genus g. Let G simple, $G \neq \mathbb{C}^*$.

$$dim\, X_G(F) = (2g-2) dim\, G$$

Hence, $dim\, X_{SU(2)}(F) = 6g - 6$ and $dim_{\mathbb{C}}\, X_{SL(2,\mathbb{C})}(F) = 6g - 6$.

Thm $\tau_1, ..., \tau_{3g-3}$ form a completely integrable system on $X_{SU(2)}(F)$ and on $X_{SL(2,\mathbb{C})}(F)$.

Rk Note that $\tau_1, ..., \tau_{3g-3} : X_{SU(2)}(F) \to \mathbb{R}$.

Proof:

(1) Poisson commuting:

Thm(Goldman)
For every classical G, loops α, β in F,

$\{\tau_\alpha, \tau_\beta\} = \sum_{c \in \alpha \cap \beta}$ expression in τ's,

where the sum is over all crossings between α and β.

Cor If α and β do not intersect then $\{\tau_\alpha, \tau_\beta\} = 0$.

(2) Alg. independence:

For $G = SL(2, \mathbb{C})$:
As an application of Teichmuller Theory, $\tau_1, ..., \tau_{3g-3}$ are alg. indep over $X_{PSL(2,\mathbb{R})}(F)$. Hence, over $X_{SL(2,\mathbb{C})}(F)$ as well.

Alg. independence of $\tau_1, ..., \tau_{3g-3}$ over $X_{SU(2)}(F)$ follows from the fact that $X_{SU(2)}(F)$ is Zariski dense in $X_{SL(2,\mathbb{C})}(F)$. $\square$

Open: Is $X_G(F)$ completely integrable for other G? (Problem: Not enough τ's.)

Thm(S.) "Analogous" complete integrability of $X_G(F)$ for all rank 2 Lie groups, eg. $SL(3,\mathbb{C})$, $SU(3)$.

Open for rank > 2.

Let us start with $G = SL(n, \mathbb{C})$.

An n-web in a surface F is an oriented graph in F whose each vertex is either an n-valent source or an n-valent sink. Loops are allowed. Crossings are allowed.

The construction of functions τ_γ on $X_G(F)$ for $G = SL(n, \mathbb{C})$ can be extended from loops to n-webs in F.

Construction of $\tau_\gamma(\rho)$ for an n-web $\gamma \subset F$:

Pull all vertices of γ to a base point b of F.

Associate to each edge, $e \in \pi_1(F, b)$ of γ
$\rho(e) \in GL(n, \mathbb{C}) \subset (\mathbb{C}^n)^* \otimes \mathbb{C}^n$.

Associate to each vertex $det : \wedge^n \mathbb{C}^n \to \mathbb{C}^n$,
i.e. $det \in \wedge^n (\mathbb{C}^n)^*$.

$\tau_\gamma(\rho)$ is the contraction of these tensors over all edges.

Lemma τ_γ is a function on $X_{SL(n,\mathbb{C})}(F)$.

Canonical webs

Lemma For every n,
# of canonical webs in $F =$
$= (g-1)(n^2-1) = \frac{1}{2} dim\, X_{SL(n,\mathbb{C})}(F)$.

This suggests that canonical webs are good candidates for a completely integrable system.

Thm (S.) (1) τ's for the canonical webs for $n = 3$ form a completely integrable system for $X_{SL(3,\mathbb{C})}(F)$.
(2) $Re\,\tau$'s for the canonical webs form a completely integrable system for $X_{SU(3)}(F)$.

Proof:

Lemma If α and β are disjoint n-webs in F, then $\{\tau_\alpha, \tau_\beta\} = 0$.

Since the canonical graphs are pairwise disjoint for $n = 3$, they Poisson commute.

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Proof of alg. independence of

$$\tau_1, ..., \tau_{8(g-1)} : X_{SL(3,\mathbb{C})}(F) \to \mathbb{C}.$$

Thm (S.) τ's give an isomorphism
$\mathbb{C}\{$3-webs in $F\}$/Kuperberg rels$\to \mathbb{C}[X_{SL(3,\mathbb{C})}(F)]$

I have a version of this theorem for all n.

Thm(S.-Westbury)
$\mathbb{C}\{$3-webs in $F\}$/Kuperberg rels
has a basis composed of all 3-webs without crossings, bi-gons and true 4-gons.

Proof uses "confluence method" for graphs developed with B. Westbury.

Using this result, it is easy to prove that monomials in canonical webs are lin. independent. Hence τ's are alg. indep. for canonical graphs.

(2) The proof of $Re\,\tau$'s being alg. indep. over $X_{SU(3)}(F)$ is technical. (Not enough to show that $X_{SU(3)}(F)$ is Zariski dense in $X_{SL(3,\mathbb{C})}(F)$.)
□

Thm. (S.) There is analogous statements for all other Lie groups of rank 2: $(P)SO(4,\mathbb{C})$, $(P)Sp(4,\mathbb{C})$, $(P)SO(5,\mathbb{C}), G_2$.

Proof:

Step 1 There is enough known about the invariant theory for rank 2 Lie groups, to describe $\mathbb{C}[X_G(F)]$ as a certain algebra of graphs in F mod local relations.

Step 2 For $rank(G) = 2$ our "confluence method" gives explicit, canonical bases of $\mathbb{C}[X_G(F)]$.

Using those bases, one shows that all monomials in canonical webs are lin. indep.

Step 3 For compact groups... □

For $SL(n,\mathbb{C})$ for $n \geq 4$,
# of canonical webs $= \frac{1}{2} dim\, X_G(F)$
but τ's for canonical webs do not Poisson commute for $n \geq 4$.