Compactification of moduli spaces of completely reducible representations

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July 2010

Settings

Counsider

- Γ any finitely generated group
- G a reductive group over a local field $\mathbb{K}$

  Example $G = \mathrm{SL}_n(\mathbb{K})$
  - $\mathbb{K} = \mathbb{R}$ (archimedean)
  - $\mathbb{K} = \mathbb{Q}_p$ (non archimedean)

The space of representations is $R = \mathrm{Hom}(\Gamma, G)$

Geometric interpretation

G acts on the associated symmetric space X

or euclidean building (non-archimedean case)

Example for $G = \mathrm{SL}_2(\mathbb{R})$, $X = \mathbb{H}^2$.

It is a nonpositively curved metric space

We are interested in the quotient space of R under G

Example For $\Gamma = \pi_1(S)$

- $G = \mathrm{SL}_2(\mathbb{R})$: Teichmüller space a cc of R/G
- $G = \mathrm{SL}_n(\mathbb{R})$: $\mathcal{T}(S)$ is included in a cc of R/G which is a cell (Hitchin)

Problem Naïve quotient R/G is not Hausdorff

(Some orbits are not closed)

Goal

1. Construct - geometrically - a good quotient space $\mathcal{X} = R//G$
2. Construct a compactification of $\mathcal{X}$
   - mimic Thurston's compactification of Teichmüller space.
   - replace the usual distance on X by a the "$\mathfrak{C}$-distance".

Outline

1. Good quotient space $\mathcal{X} = R//G$ via complete reducibility
2. The $\mathfrak{C}$-distance on X
3. Compactification of $\mathcal{X} = R//G$

Outline

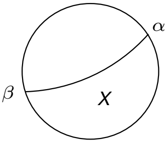
1. Good quotient space $\mathcal{X} = R//G$ via complete reducibility
   - complete reducibility in X
   - Good quotient
2. The $\mathfrak{C}$-distance on X
3. Compactification of $\mathcal{X} = R//G$

How to construct a good quotient ?

Remarks

- Closed orbits does not imply Hausdorff quotient in general
  Example : $\Gamma = \mathbb{Z}$, $X = \mathbb{R}^2$ and $G = \mathrm{Isom}\,(X)$
- Classical methods : - Algebraic geometry (GIT)
  - Symplectic geometry, moment map
  $\rightsquigarrow$ good quotient from closed orbits
- We give a geometric definition and proof (via action on X)

Complete reducibility on Hadamard spaces



For two points α, β in $\partial_\infty X$
opposite= joined by a geodesic in X

Definition (Serre)

$\rho : \Gamma \to G$ is completely reducible (cr) if,
if ρ fixes some point in $\partial_\infty X$
then ρ also fixes an opposite point in $\partial_\infty X$

For $G = \mathrm{SL}_n\mathbb{R}$ and $X = \mathrm{SL}_n/\mathrm{SO}(n)$: ρ cr $\Leftrightarrow$ ρ semisimple on $\mathbb{R}^n$

Geometric characterizations of cr

To a representation $\rho : \Gamma \longrightarrow G$ we associate the convex function on X (displacement function)

$$\begin{array}{rccl} d_\rho : & X & \to & \mathbb{R}^+ \\ & x & \mapsto & \sqrt{\sum_{s\in S} d(x,\rho(s)x)^2} \end{array}$$

Theorem

Suppose $\mathbb{K} = \mathbb{R}$. For $\rho : \Gamma \longrightarrow G$, tfae

(i) *ρ is completely reducible*

(ii) *ρ stabilize a closed convex $Y \subset X$, and $Y = Y_0 \times Y'$ with Y_0 translated and no fixed point in $\partial_\infty Y'$.*

(iii) *d_ρ has a minimal value*

Remarks

- $(iii) \Rightarrow (i)$ false in a tree
- related to existence of harmonic maps

Good quotient of $\mathrm{R} = \mathrm{Hom}(\Gamma, G)$

Theorem

- *Every orbit contains in it closure a unique cr orbit*
- *The corresponding "semisimplification" map $\pi : \mathrm{R} \longrightarrow \mathrm{R}_{cr}/G$ is the biggest Hausdorff quotient.*

This is a classical result from GIT for $\mathbb{K} = \mathbb{C}$
$\mathbb{K} = \mathbb{R}$: Luna, Richardson-Slodowy (moment map)
Local fiels of char 0 : Bremigan 94

We give a new proof : direct, geometric, valid for all local fields (including the new case of char > 0)

Outline

1. Good quotient space $\mathcal{X} = R//G$ via complete reducibility

2. The $\mathfrak{C}$-distance on X
   - Definition
   - $\mathfrak{C}$-length of an isometry

3. Compactification of $\mathcal{X} = R//G$

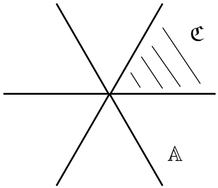
The $\mathfrak{C}$-distance on X

Fix $x_0 \in X$, a maximal flat $\mathbb{A}$ of X and $\mathfrak{C}$ a Weyl chamber in $\mathbb{A}$.

Example

For $G = \mathrm{SL}_n$

- $\mathbb{A} = \{(\lambda_1, \ldots, \lambda_n),\ \sum \lambda_i = 0\}$
- $\mathfrak{C} = \{(\lambda_1 \geq \ldots \geq \lambda_n)\}$



Fact

$\overline{\mathfrak{C}}$ is a strict fundamental domain for $K = \mathrm{Stab}_G x_0$ in X.

Definition

The $\mathfrak{C}$-distance is the corresponding projection

$$\delta : X \times X \longrightarrow \overline{\mathfrak{C}}$$

- refines usual distance (equal in rk 1)

Properties of the $\mathfrak{C}$-distance

Remark

The $\mathfrak{C}$-distance δ satifies remarkable distance-like properties, notably

- triangular inequalities
- convexity properties

with respects to a suitable partial order in $\mathbb{A}$
(work in progress)

$\mathfrak{C}$-length of an isometry

Define the $\mathfrak{C}$-length of $g \in G$ by

$$\ell^{\mathfrak{C}}(g) = \inf_{x \in X} \delta(x, gx)$$

- "inf" is for the partial order on $\mathbb{A}$
- The $\mathfrak{C}$-length refines usual translation length (equal in rk 1)
- Algebraically it is Jordan projection.
- For $G = \mathrm{SL}_n$, it's $\ell^{\mathfrak{C}}(g) = (\log|\lambda_1(g)| \geq \ldots \geq \log|\lambda_n(g)|)$ (eigenvalues)

Outline

1. Good quotient space $\mathcal{X} = R//G$ via complete reducibility

2. The $\mathfrak{C}$-distance on X

3. Compactification of $\mathcal{X} = R//G$
   - Construction
   - On the proof

Compactification of $\mathcal{X} = R//G$

Let $\mathcal{X} = R//G = \mathrm{R}_{cr}/G$ the biggest Hausdorff quotient of $R = \mathrm{Hom}(\Gamma, G)$ under G.

- For $\rho : \Gamma \longrightarrow G$ the (marked) $\mathfrak{C}$-length spectrum is

$$\ell^{\mathfrak{C}} \circ \rho \in \overline{\mathfrak{C}}^{\Gamma}$$

- Projectivization : $\mathbb{P}\overline{\mathfrak{C}}^{\Gamma} = (\overline{\mathfrak{C}}^{\Gamma} - \{0\})/\sim$

Theorem (Compactification)

The projectivized $\mathfrak{C}$-length spectrum

$$\begin{array}{rccc} \mathbb{P}\mathcal{L}^{\mathfrak{C}} : & \mathcal{X} & \to & \mathbb{P}\overline{\mathfrak{C}}^{\Gamma} \\ & [\rho] & \mapsto & [\ell^{\mathfrak{C}} \circ \rho] \end{array}$$

induces a compactification $\widetilde{\mathcal{X}}$ of $\mathcal{X}$, which boundary $\partial_\infty \widetilde{\mathcal{X}} - \mathcal{X}$ consists of points $[w] \in\subset \mathbb{P}\overline{\mathfrak{C}}^{\Gamma}$ that are projectivized $\mathfrak{C}$-length spectra of actions of Γ on euclidean buildings.

Description (convergent sequences)

$$[\rho_i]_{i\in\mathbb{N}} \to [w] \in \partial_\infty \widetilde{\mathcal{X}} \text{ iff } \begin{cases} [\rho_i] \text{ gets out of any compact} \\ [\ell^{\mathfrak{C}} \circ \rho_i] \to [w] \text{ in } \mathbb{P}\overline{\mathfrak{C}}^{\Gamma} \end{cases}$$

Remarks

- Extends
  - $G = \mathrm{SL}_2(\mathbb{R})$:Thurston's compactification of $\mathrm{Teich}(S)$
  - $\mathrm{rk}(G) = 1$: Morgan-Shalen, Bestvina, Paulin
- Natural action of $\mathrm{Out}(\Gamma)$ on $\widetilde{\mathcal{X}}$
- Gives compactifications for generalized Teichmüller spaces
- For boundary points
  - buildings involved are nondiscrete (extend real trees)
  - no global fixed point
  - the action comes from $\rho : \Gamma \longrightarrow G(\mathbb{F})$, where $\mathbb{F}$ is a non archimedean field.

Ingredients of the proof

- Renormalize $\mathcal{L}^{\mathfrak{C}}$ by minimal displacement

$$\lambda(\rho) = \inf_X d_\rho = \inf_{x \in X} \sqrt{\sum_{s \in S} d(x, \rho(s)x)^2}$$

  to stay in a compact of $\overline{\mathfrak{C}}^\Gamma$.
- Main point : Show that 0 is not in the closure of $\frac{1}{\lambda}\mathcal{L}^{\mathfrak{C}}(\mathcal{X} - C)$ for suitable compact C.
- Use asymptotic cones (Gromov) to get actions on buildings
  - $\mathfrak{C}$-length pass continuously to asymptotic cones
- Length spectrum of actions on euclidean buildings with no global fixed point are non zero

Thank you for your attention.

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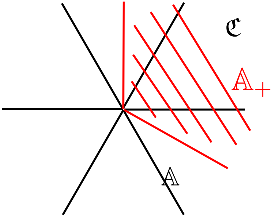
the partial order in $\mathbb{A}$

The partial order in $\mathbb{A}$
$v \geq_{\mathbb{A}} 0$ in $\mathbb{A}$
iff $(v, u) \geq 0$ for all $v \in \mathfrak{C}$



Positive cone

Denote by $\Theta : \mathbb{A} \longrightarrow \overline{\mathfrak{C}}$ the canonical projection.