Pseudo-Anosovs with small dilatation and the Dehn fillings of the magic manifold

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Pseudo-Anosov (1)

$\Sigma = \Sigma_{g,n}$; orientable surface of genus g with n punctures

$\mathrm{Mod}(\Sigma)$; the mapping class group of Σ.

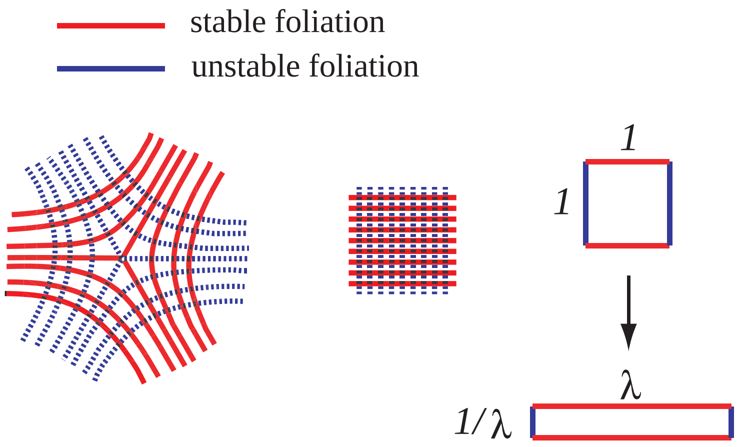
We focus on pseudo-Anosov elements of $\mathrm{Mod}(\Sigma)$.

Pseudo-Anosov (2)

- A homeomorphism $\Phi : \Sigma \to \Sigma$ is *pseudo-Anosov* if
  $^\exists \lambda = \lambda(\Phi) > 1$ called the *dilatation of* Φ, and
  $^\exists \mathcal{F}^s, \mathcal{F}^u$; a pair of transverse measured foliations such that

$$\Phi(\mathcal{F}^s) = \frac{1}{\lambda}\mathcal{F}^s \text{ and } \Phi(\mathcal{F}^u) = \lambda \mathcal{F}^u.$$

$\mathcal{F}^s$ and $\mathcal{F}^u$ are called the *stable* and *unstable foliation*.



Dilatation, Entropy

The mapping class $\phi = [\Phi] \in \mathrm{Mod}(\Sigma)$ containing a pseudo-Anosov homeo Φ is called pseudo-Anosov.

- $\lambda(\phi) := \lambda(\Phi) > 1$; dilatation of ϕ

- $\log \lambda(\phi)$; entropy of ϕ

- $|\chi(\Sigma)|\lambda(\phi)$; normalized dilatation
- $|\chi(\Sigma)| \log(\lambda(\phi))$; normalized entropy

Minimal dilatation (1)

Fix a surface $\Sigma_{g,n}$.

$$Spec(\Sigma_{g,n}) := \{\lambda(\phi) \mid \text{pseudo-Anosov } \phi \in \mathrm{Mod}(\Sigma_{g,n})\}.$$

★ There exists a minimum of $Spec(\Sigma_{g,n})$ (Ivanov)

$\delta_{g,n} := \min Spec(\Sigma_{g,n})$

$\delta_g := \delta_{g,0}$.

Minimal dilatation (2)

Question 1. *What is the value δ_g for $g \geq 3$?*

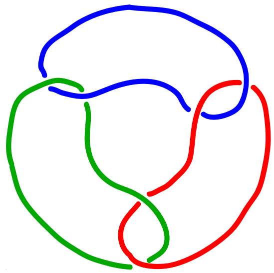
★ $\log \delta_g \asymp 1/g$ (Penner, 1991)

Question 2 (McMullen, 2000)**.**

- *Does* $\lim_{g \to \infty} g \log \delta_g$ *exist?*
  $\iff$ *Does* $\lim_{g \to \infty} \underbrace{|\chi(\Sigma_{g,0})| \log \delta_g}_{\text{minimal normalized entropy of genus } g}$ *exist?*
- *What is its value?*

Magic manifold N

- $N := S^3 \setminus$ (3 chain link)
- hyperbolic, fibered, vol$(N) = 5.33348...$, the smallest known volume among orientable hyperbolic 3-manifolds with 3 cusps.



We study pseudo-Anosovs which occur as the monodromies on fibers for Dehn fillings of N.

Small dilatation pseudo-Anosovs (1)

For $P > 1$, define

$$\Psi_P := \{\text{pseudo-Anosov } \Phi : \Sigma \to \Sigma \; ; \; |\chi(\Sigma)| \log \lambda(\Phi) \leq \log P\}.$$

Elements of Ψ_P are called the small dilatation pseudo-Anosovs.

For P sufficiently large, (e.g, $P \geq 2 + \sqrt{3}$),

$$\Psi_P \supset \{\Phi_g : \Sigma_{g,0} \to \Sigma_{g,0}\}_{g \geq 2}$$

(Ψ_P is an infinite set)

Theorem 1 (Farb-Leininger-Margalit)**.** *For any $P > 1$, there exist finite many hyperbolic, fibered 3-mfds $M_1, M_2, \cdots, M_r$ such that any $\Phi \in \Psi_P$ occurs as the monodromy of a Dehn filling of one of the M_k.*

Small dilatation pseudo-Anosovs (2)

★ (K-Takasawa, 2009) For each $n = 3, \cdots, 8$ (resp. $n \geq 9$), the pseudo-Anosov homeo. on an n-punctured disk with the smallest dilatation (resp. smallest known dilatation) occurs as the monodromy on a fiber for a Dehn filling of N.

(Independently, this is also established by Venzke.)

Results

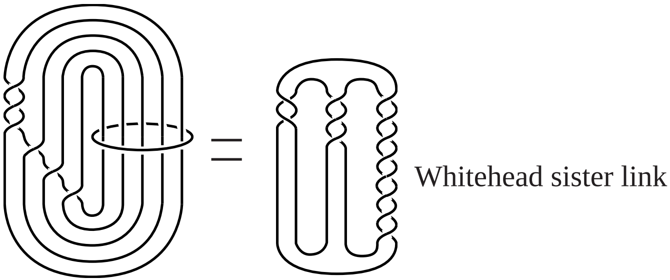
For $r \in \mathbb{Q}$, let $N(r)$ be the Dehn filling of N along the slope r.

Main Theorem. (K-Takasawa) Let $r \in \{\frac{-3}{2}, \frac{-1}{2}, 2\}$. Consider the Dehn filling $N(r)$. For any $g \geq 3$, there exists a monodromy $\Phi_g(r) : \Sigma_g \to \Sigma_g$ on a closed fiber of genus g for a Dehn filling of $N(r)$ such that

$$\lim_{g \to \infty} g \log \lambda(\Phi_g(r)) = \log(\tfrac{3+\sqrt{5}}{2}) = \log(1 + \text{golden ratio}).$$

In particular

$$\lim_{g \to \infty} \sup g \log \delta_g \leq \log(\tfrac{3+\sqrt{5}}{2}).$$



- Case $r = \frac{-1}{2}$; established by Hironaka
- Case $r = \frac{-3}{2}$; established by Aaber-Dunfield

$N(\frac{-3}{2}) \simeq (-2, 3, 8)$-pretzel link (= Whitehead sister link) exterior

An upper bound of δ_g given by K-Takasawa and Hironaka

(K-Takasawa)	(Hironaka)
	$\delta_3 \leq 1.401268,\ N(\frac{-1}{2}, *, *)$
	$\delta_4 \leq 1.261230,\ N(\frac{-1}{2}, *, *)$
$\delta_5 \leq 1.148794,\ N(\frac{-3}{2}, *, *)$	
$\delta_6 \leq 1.128760,\ N(\frac{-3}{2}, *, *)$	
$\delta_7 \leq 1.115481,\ N(\frac{-3}{2}, *, *)$	
$\delta_8 \leq 1.104039,\ N(\frac{-4}{3}, \frac{-25}{17}, \frac{-5}{1})$	
$\delta_9 \leq 1.092824,\ N(\frac{-3}{2}, *, *)$	
$\delta_{10} \leq 1.083766,\ N(\frac{-3}{2}, *, *)$	
$\delta_{11} \leq 1.077045,\ N(\frac{-3}{2}, *, *)$	
$\delta_{12} \leq 1.072664,\ N(\frac{-3}{2}, *, *)$	
$\delta_{13} \leq 1.071692,\ N(\frac{-29}{27}, \frac{-5}{3}, \frac{-6}{1})$	
$\delta_{14} \leq 1.062987,\ N(\frac{-3}{2}, *, *)$	
$\delta_{15} \leq 1.058335,\ N(\frac{-3}{2}, *, *)$	
$\delta_{16} \leq 1.054998,\ N(\frac{-3}{2}, *, *)$	
$\delta_{17} \leq 1.052214,\ N(\frac{-3}{2}, *, *)$	
	$\delta_{18} \leq 1.052540,\ N(\frac{-1}{2}, *, *)$
$\delta_{19} \leq 1.047084,\ N(\frac{-3}{2}, *, *)$	

Pseudo-Anosovs with orientable foliations

$\delta_g^+ := \min\{\lambda(\phi) \mid \phi \in \mathrm{Mod}(\Sigma_{g,0})$ with orientable (un)stable foliation$\}$.

Clearly $\delta_g \leq \delta_g^+$.

★$\delta_2 = \delta_2^+ \approx 1.722083$ (Zhirov, Cho-Ham)

Minimal dilatation δ_g^+

Theorem 2 (K-Takasawa, Aaber-Dunfield).

$$\delta_7^+ = \lambda_{(9,2)} \approx 1.11548,$$

where $\lambda_{(k,\ell)}$ *is a unique real root greater* 1 *of the polynomial*

$$f_{(k,\ell)}(t) = t^{2k} - t^{k+\ell} - t^k - t^{k-\ell} + 1 \ \ \textit{for}\ k > 0,\ \ -k < \ell < k$$

- $\delta_2^+ = \lambda_{(2,1)} \approx 1.72208$ (Zhirov, Cho-Ham)
- $\delta_3^+ = \lambda_{(3,1)} = \lambda_{(4,3)} \approx 1.40127$ (Lanneau-Thiffeault)
- $\delta_4^+ = \lambda_{(4,1)} \approx 1.28064$ (Lanneau-Thiffeault)
- $\delta_5^+ = \lambda_{(6,1)} = \lambda_{(7,4)} \approx 1.17628$ (Lanneau-Thiffeault)
- $\delta_8^+ = \lambda_{(8,1)} \approx 1.12876$ (Hironaka)

Proposition 1 (K-Takasawa, Aaber-Dunfield).

$$\delta_5 < \delta_5^+.$$

Question on minimal dilatation δ_g^+

Question 3 (Lanneau-Thiffeault). *For g even, is δ_g^+ equal to $\lambda_{(g,1)}$?*

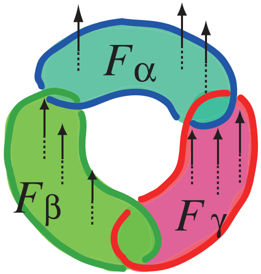
★$\delta_2^+ = \lambda_{(2,1)}$ (Zhirov).

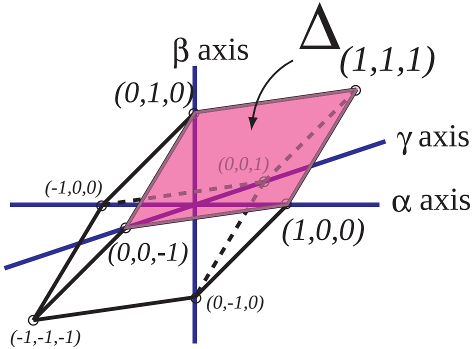
★$\delta_4^+ = \lambda_{(4,1)}$ and $\delta_6^+ \geq \lambda_{(6,1)}$ (Lanneau-Thiffeault).

Proposition 2 (K-Takasawa). *If Question 3 is true, then $\delta_g^+ < \delta_{g+1}^+$ for $g \equiv 1, 5, 7, 9 \pmod{10}$ and $g \geq 7$. In particular $\delta_7^+ < \delta_8^+$.*

Background, Idea of Main Theorem

- Thurston norm $X_T : H_2(N, \partial N; \mathbb{R}) \to \mathbb{R}$.





- McMullen's Teichmüler polynomial P_Δ on a fiber face Δ.
- Fact: $\exists$ a unique ray on $int(C_\Delta)$ which realizes

$$\min\{X_T(a) \cdot \log \lambda(a) \mid a \in int(C_\Delta)\}$$

What we need to show:

- find that minimal ray.
- find a nice sequence of fibers $\{F_n\}$ so that the ray of homology classes $\{[F_n]\}$ goes to the minimal ray as n goes to ∞.

The magic "magic manifold"

★ There are pseudo-Anosov monodromies for Dehn fillings of N realizing

$$\delta_2,$$
$$\delta_n^+ \text{ for } n = 2, 3, 4, 5, 7, 8,$$
$$\delta(D_n) \text{ for } n = 3, 4, \cdots, 8.$$

Question 4. *Let $g \geq 3$ and $n \geq 9$.*

(1) *Is there a pseudo-Anosov monodromy on a closed fiber Σ_g realizing δ_g (resp. δ_g^+) for some Dehn filling of N?*

(2) *Is there a pseudo-Anosov monodromy on a fiber D_n realizing $\delta(D_n)$ for some Dehn filling of N?*