# Polynomial automorphisms of the Fricke characters of a free group

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# The model

For a rank-$n$ free group $F_n$, call

$$V_{F_n} = Hom(F_n, SL(2,\mathbb{C}))//SL(2,\mathbb{C})$$

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$Out(F_n) = Aut(F_n)/Inn(F_n)$ acts on $V_{F_n}$. We study the dynamics of this action in terms of how individual elements act.

Today, we discuss a model to facilitate this study.

# Fricke Characters

Order a basis for $F_n = \langle A, B, C, \ldots \rangle$ so that $A > B > C > \ldots$ and extend to all basic words: $\alpha \in F_n$ is *basic* if

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Also, for $\alpha \in F_n$, call

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the $SL(2,\mathbb{C})$-character of $\alpha$. Call $\mathcal{H}_n = \left\{ tr_\beta \middle| \beta \text{ is basic in } F_n \right\}$ the *Horowitz generating set*.

**Theorem (Horowitz)**

*For any $\alpha \in F_n$, $tr_\alpha \in \mathbb{Z}[\mathcal{H}_n]$.*

# Fricke Characters

For $n > 2$, $\exists$ a nontrivial ideal $\mathcal{I}_n \subset \mathbb{Z}[\mathcal{H}_n]$ of "trace relations": Complicated versions of Cayley-Ham. form of characteristic poly. of elements in $SL(2,\mathbb{C})$:

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Via the CS-evaluation map

$$T : Hom(F_n, SL(2,\mathbb{C})) \to \mathbb{C}^{2^n-1}, \quad T(\phi) = (\mathcal{H}_n(\phi)),$$

$V_{F_n} = T\left(Hom(F_n, SL(2,\mathbb{C}))\right) \subset \mathbb{C}^{2^n-1}$ ($\mathcal{H}_n$ called trace coordinates) as the common 0-set of $\mathcal{I}_n$.

**Example (Gonz.-Acuña,Mont.-Am.)**

$V_{F_2} \cong \mathbb{C}^3$ and for $n \geq 2$, $V_{F_n} \subset \mathbb{C}^{2^n-1}$ is $3n-3$-dim. $\mathcal{I}_3$ is principal, but a basis for $\mathcal{I}_4$ uses 12 gens. to cut the 9-dim. $V_{F_4}$ out of $\mathbb{C}^{15}$.

# Automorphisms of $V_{F_n}$

To describe the $Out(F_n)$ action, present $F_n = \langle A_1, A_2, A_3, \ldots \rangle$, so that

$$U : A_1 \mapsto A_1 A_2 \quad P : \begin{array}{l} A_1 \mapsto A_2 \\ A_2 \mapsto A_1 \end{array} \quad Q : \begin{array}{l} A_i \mapsto A_{i+1} \\ A_n \mapsto A_1 \end{array} \quad \sigma : A_1 \mapsto A_1^{-1}$$

are the 4 Nielsen generators of $Aut(F_n)$.

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Interp. as outer autos., their action on $\mathcal{H}_n$ specifies the action on $\mathbb{Z}[\mathcal{H}_n]/\mathcal{I}_n$. Individually, we can extend each to an auto of $\mathbb{Z}[\mathcal{H}_n]$. But not uniquely for $n > 2$.

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Via the eval. map, each of $U, P, Q, \sigma$ yields a poly. auto. of $\mathbb{C}^{2^n-1}$. The (right) action is given by

$$\text{For } \theta \in Out(F_n), \quad \widehat{\theta} : \mathbb{C}^{2^n-1} \to \mathbb{C}^{2^n-1}, \quad tr_{\theta(\alpha)} = (tr_\alpha)\,\widehat{\theta}.$$

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**Example ($n = 3 : V_{F_3} \in \mathbb{C}^7$ with coords. $(tr_A, tr_B, \ldots, tr_{ABC})$)**

$(tr_{ABC})\,\widehat{U} = tr_{ABBC} = tr_{CABB} = tr_{CAB}\,tr_B - tr_{CABB^{-1}} = tr_{ABC}\,tr_B - tr_{AC}.$

# Individual lifts of generators of $Out(F_n)$

**Example ($n = 2$: Let $(tr_A, tr_B, tr_{AB}) = (x, y, z)$)**

$$\widehat{U} : \begin{array}{rcl} x & \mapsto & z \\ y & \mapsto & y \\ z & \mapsto & yz - x \end{array} \qquad \widehat{P}, \widehat{Q} : \begin{array}{rcl} x & \mapsto & y \\ y & \mapsto & x \\ z & \mapsto & z \end{array} \qquad \widehat{\sigma} : \begin{array}{rcl} x & \mapsto & x \\ y & \mapsto & y \\ z & \mapsto & xy - z \end{array}$$

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# Automorphisms of $V_{F_n}$

Q. Does $Out(F_n)$-action on $V_{F_n}$ extend to ambient space $\mathbb{C}^{2^n-1}$? (Does action on quotient ring $\mathbb{Z}[\mathcal{H}_n]/\mathcal{I}_n$ extend to an action on $\mathbb{Z}[\mathcal{H}_n]$?)

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A. No, in general. "Lift" each of $U, P, Q, \sigma \in Out(F_n)$ to the poly autos $\widehat{U}, \widehat{P}, \widehat{Q}, \widehat{\sigma}$ of $\mathbb{C}^{2^n-1}$ and call $POut(F_n) = \langle \widehat{U}, \widehat{P}, \widehat{Q}, \widehat{\sigma} \rangle$.

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**Theorem (McCool)**

*This "lifting" induces an epimorphism*

$$\Phi_n : POut(F_n) \to Out(F_n)$$

*which is an isomorphism only for $n \leq 3$.*

# Structure of $POut(F_n)$

- For $n = 2, 3$, the $Out(F_n)$ on $V_{F_n}$ extends into ambient space.
- McCool shows this by producing a member of $ker\Phi_n$, $n > 3$.
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**Theorem (Brown)**

*For $n \in \mathbb{N}$, $POut(F_n)$ acts on $\mathbb{C}^{2^n-1}$ as volume preserving integer poly autos which leave invariant $V_{F_n}$ and restrict to volume preserving $Out(F_n)$ action on $V_{F_n}$.*

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**Theorem (Brown)**

*Let $F_n = \pi_1(S)$ for a compact surface $S$, and call $PMCG(S)$ the pre-image of $MCG(S)$ under $\Phi_n$. Then $PMCG(S)$ acts as unit Jacobian polynomial automorphisms on $\mathbb{C}^{2^n-1}$ which restrict to the $MCG(S)$ action on $V_{F_n}$.*

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In the Nielsen presentation, there are 18 non-inner relations. Of these, 3 do not evaluate to 1 in $POut(F_n)$:

$$R_1 = \left[ Q^{-1}PQPQ^{-1}, U^{-1} \right], \quad R_2 = U^{-1}QU^{-1}Q^{-1}U \left( QUQ^{-1}P \right)^2, \quad R_3 = (PQ)^{n-1}.$$

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# Generators of $ker\Phi_4$

Let $F_4 = \langle A, B, C, D \rangle$, so that

$$\begin{aligned} \mathcal{H}_4 &= \{tr_A, tr_B, tr_C, tr_D, tr_{AB}, \ldots, tr_{BCD}, tr_{ABCD}\} \\ &= \{l, m, n, o, p, \ldots, y, z\} \end{aligned}$$

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as coordinates of $\mathbb{C}^{15}$. Then

$$\widehat{K}_1 : \begin{array}{rcl} l & \mapsto & l \\ m & \mapsto & m \\ n & \mapsto & n \\ o & \mapsto & o \\ p & \mapsto & p \\ q & \mapsto & q \\ r & \mapsto & r \\ s & \mapsto & s \\ t & \mapsto & t \\ u & \mapsto & u \\ v & \mapsto & v \\ w & \mapsto & -lmo - lny + lsu \\ & & +lt + mr + nz + op \\ & & +qy - sx - uv - w \\ x & \mapsto & x \\ y & \mapsto & y \\ z & \mapsto & z \end{array} \qquad \widehat{K}_2 : \begin{array}{rcl} l & \mapsto & l \\ m & \mapsto & m \\ n & \mapsto & n \\ o & \mapsto & o \\ p & \mapsto & p \\ q & \mapsto & q \\ r & \mapsto & r \\ s & \mapsto & s \\ t & \mapsto & t \\ u & \mapsto & u \\ v & \mapsto & v \\ w & \mapsto & w \\ x & \mapsto & -lno - mov + ops \\ & & +lu + mz + nr + oq \\ & & -py - sw + tv - x \\ y & \mapsto & y \\ z & \mapsto & z \end{array} \qquad \widehat{K}_3 : \begin{array}{rcl} l & \mapsto & l \\ m & \mapsto & m \\ n & \mapsto & n \\ o & \mapsto & o \\ p & \mapsto & p \\ q & \mapsto & q \\ r & \mapsto & r \\ s & \mapsto & s \\ t & \mapsto & t \\ u & \mapsto & u \\ v & \mapsto & v \\ w & \mapsto & w \\ x & \mapsto & x \\ y & \mapsto & y \\ z & \mapsto & lmno - lmu - los \\ & & -mnr - nop + ly \\ & & +mx + nw + ov \\ & & +pu + rs - qt - z \end{array} \tag{1}$$

# Structure of $ker\Phi_n$

Showing there are no other relations among the $\widehat{K}_i$, $i = 1, 2, 3$, is tricky, but amounts to constructing the geometric representation that defines the Coxeter group explicitly. This is done by restricting these autos to an appropriate invariant vector space in $\mathbb{C}^{15}$.

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**Theorem (Brown)**

*For $n > 4$, $\ker \Phi_n$ is rank-3 with all generators of infinite order.*

# Notes

## Remark (Other embeddings)

Constructing $POut(F_n)$ via other CS-embeddings into $\mathbb{C}^m$ does not lead to an action by volume preserving automorphisms.

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## Remark (New reps of subgroups of $Out(F_n)$ and $MCG(S)$)

Let $p \in V_{F_n}$ be periodic under $POut(F_n)$. Then $POut(F_n)_p$ is a finite index subgroup.

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## Remark (New reps of subgroups of $Out(F_n)$ and $MCG(S)$)

Let $p \in V_{F_n}$ be periodic under $POut(F_n)$. Then $POut(F_n)_p$ is a finite index subgroup. The resulting tangent linear representation is a rep into $SL(2^n - 1, \mathbb{C})$ which restricts to a rep of $Out(F_n)_p$ into $SL(3n-3, \mathbb{C})$ (at least off of the singular set).

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*A covering* $\rho : S \to T$ *induces* $\rho_* : \pi_1(S) \hookrightarrow \pi_1(T)$ *which induces* $\overline{\rho} : V_{\pi_1(T)} \hookrightarrow V_{\pi_1(S)}$, *an embedding.*

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*A covering* $\rho : S \to T$ *induces* $\rho_* : \pi_1(S) \hookrightarrow \pi_1(T)$ *which induces* $\overline{\rho} : V_{\pi_1(T)} \hookrightarrow V_{\pi_1(S)}$, *an embedding.*
*Mapping classes which commute with* $\rho$ *agree on* $\overline{p}\left(V_{\pi_1(T)}\right) \subset V_{\pi_1(S)}$.

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- *Can use POut to calculate the dynamical type along the normal directions.*

# Notes

## Application (More Dynamics)

*[Bellon-Viallet] For a polynomial endomorphism $f$ of $C^m$, define the algebraic entropy*

$$d_f = \log \lim_{n\to\infty} (deg f^n)^{\frac{1}{n}}$$

*as the asympt exp growth rate of the degree.*

- *Measures the growth in per. points of $f$ and approx. top. entropy.*
- *Calculated this for $SU(2)$-char. var. of a punct. torus (some real points of $V_{F_2}$. [TAMS'06]*
- *Hadari: The algebraic entropy of an element in $POut(F_n)$ is the same as that of the corresponding element in $Out(F_n)$.*