Local Connectivity of Deformation spaces of Kleinian groups

Aaron Magid

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30 July 2010

- Review the definition of $AH(M)$
- Topology of the interior
- Bumponomics and the failure of local connectivity
- A local model for $AH(M)$

Let M be a compact, orientable 3-manifold.

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If $P \subset \partial M$ is a collection of annuli and tori

$$AH(M,P) = \{\rho \in AH(M) \mid \rho(g) \text{ parabolic for all } g \in P\}$$

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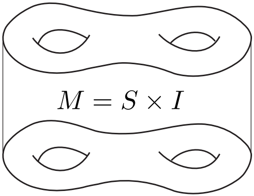
$\rho \in AH(M)$



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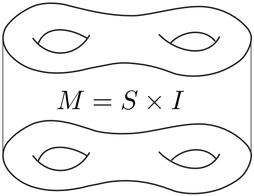

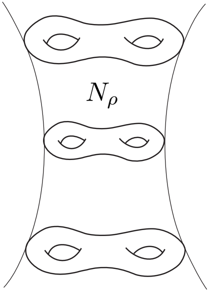

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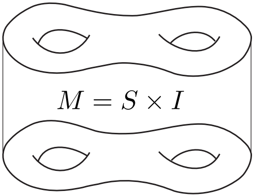
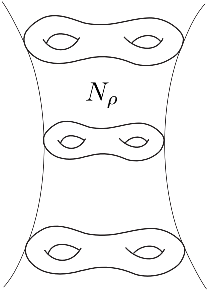


$$\xrightarrow[\substack{\text{homotopy}\\ \text{equivalence}}]{f_\rho}$$



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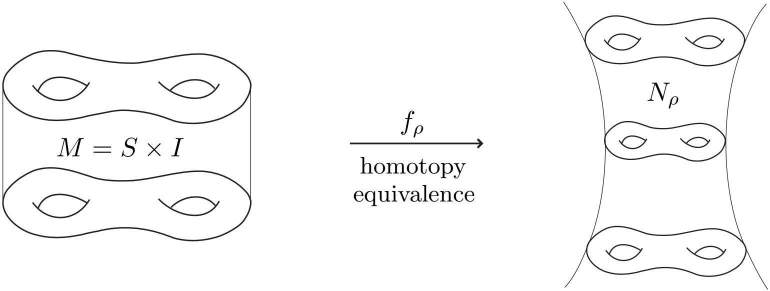

$AH(M)$ is the set of marked hyperbolic 3-manifolds homotopy equivalent to M

Theorem (Ahlfors, Bers, Kra, Marden, Maskit, Sullivan, Thurston)

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Example:

$$int(AH(S \times I)) \cong \mathcal{T}(S) \times \mathcal{T}(S)$$

Theorem (Brock, Bromberg, Kim, Kleineidam, Lecuire, Namazi, Ohshika, Souto, Thurston)

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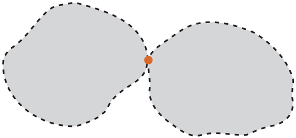

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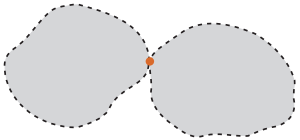
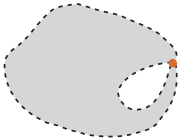
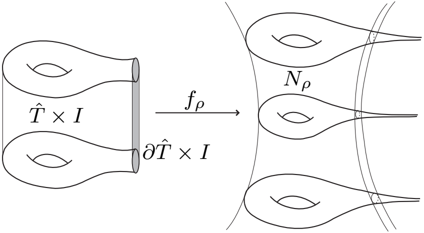


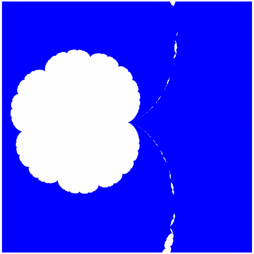
The interior of $AH(S \times I)$ self-bumps:
(McMullen, Bromberg, Holt)



Theorem (Bromberg)

Let $\hat{T}$ denote the punctured torus. Then $AH(\hat{T} \times I, \partial\hat{T} \times I)$ is not locally connected.





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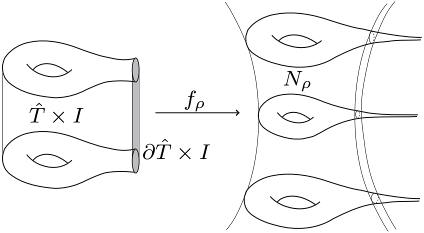

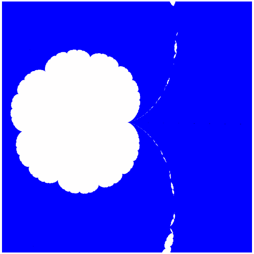

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- Depends on Minsky's classification of punctured torus groups

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For any closed genus $g \geq 2$ surface S, $AH(S \times I)$ is not locally connected.

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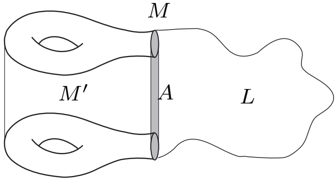
Let M be a hyperbolizable 3-manifold with incompressible boundary containing a primitive essential annulus A and suppose $(\hat{T} \times I, \partial\hat{T} \times I)$ is pared homeomorphic to one of the components (M', A) of $M - A$. Then $AH(M)$ is not locally connected.



$$\begin{array}{c} AH(M) \\ \Big\downarrow \text{want to use } \rho \mapsto \rho|_{\pi_1(\hat{T})} \\ AH(\hat{T}, \partial\hat{T}) \end{array}$$

$$AH(M)$$

$$\mathcal{A}_{\hat{T}} \xrightarrow{\Phi} AH(\hat{T}, \partial\hat{T})$$

- Local model for $AH(\hat{T}, \partial\hat{T})$ [Bromberg]

$$\mathcal{A}_M \xrightarrow{\quad\Phi\quad} AH(M)$$

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- Local model for $AH(\hat{T}, \partial\hat{T})$ [Bromberg]
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  $\mathcal{A}_M$ not locally connected since $\mathcal{A}_{\hat{T}}$ not locally connected

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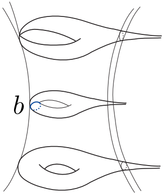
- Complex length estimates from Filling Theorem show $\overline{\Phi(\mathcal{A}_M)}$ not locally connected. Density implies $AH(M)$ not locally connected.

Local model for $AH(\hat{T}, \partial\hat{T})$



Given σ with extra parabolic

$$\sigma(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

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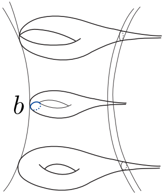
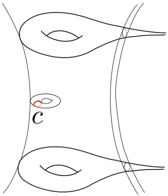


$\xrightarrow{\text{extend}}$



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construct manifold with rank 2 cusp

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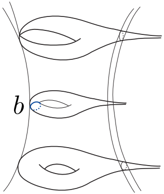

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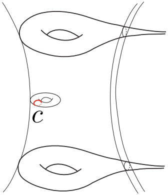

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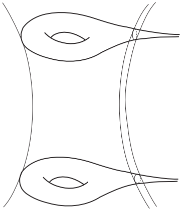
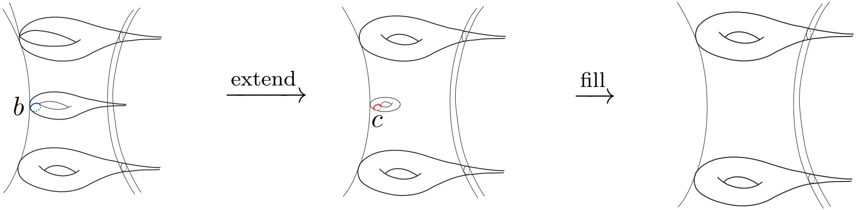


fill along c

Local model for $AH(\hat{T}, \partial\hat{T})$



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$$\Phi(\sigma, w) = \begin{cases} \text{filling of } \mathbb{H}^3/\sigma_w & \text{if } w \neq \infty \\ \sigma & \text{if } w = \infty \end{cases}$$

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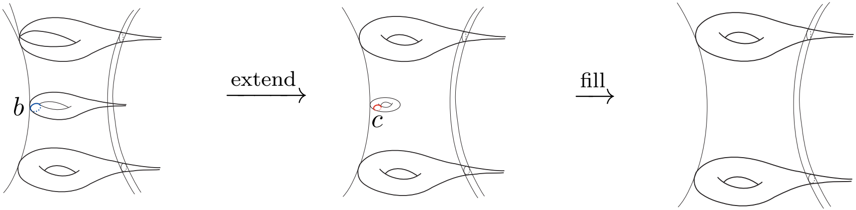

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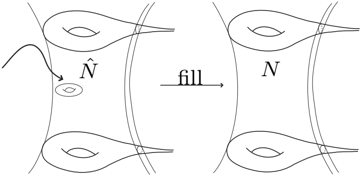
Theorem (Bromberg)

Φ *extends to a local homeomorphism* $\overline{\mathcal{A}_{\hat{T}}} \to AH(\hat{T}, \partial \hat{T})$.

Hyperbolic Dehn filling

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$$



Theorem (Hodgson-Kerckhoff, Bromberg, Brock-Bromberg, M)

Let $L > 1$, $\varepsilon > 0$. There exists K such that if $|w| > K$, then

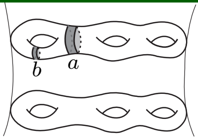
- *the hyperbolic Dehn filling of $\hat{N}$ exists*
- *the complex length, $l + i\theta$, of the core curve, γ, of the solid filling torus satisfies*

$$\left| l - \frac{4\pi Im(w)}{|w|^2} \right| \leq \frac{16(2\pi)^3 (Im(w))^2}{|w|^4} \qquad \left| \theta - \frac{4\pi Re(w)}{|w|^2} \right| \leq \frac{10(2\pi)^3 (Im(w))^2}{|w|^4}$$

- *there exists an L-biLipschitz diffeomorphism*

$$\hat{N} - \{\varepsilon - \textit{thin part about cusp}\} \to N - \{\varepsilon - \textit{thin part about } \gamma\}$$

Local model for (most of) $AH(M)$



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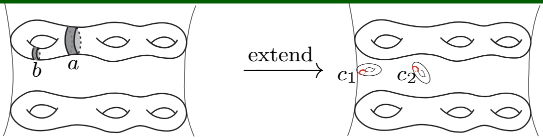

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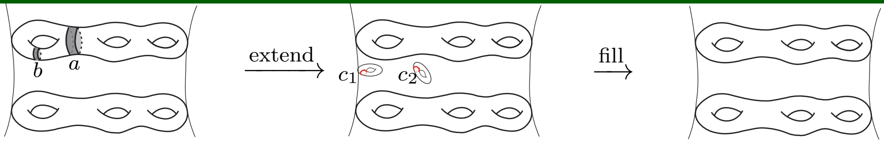

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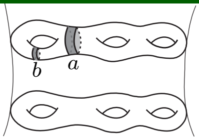

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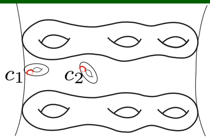

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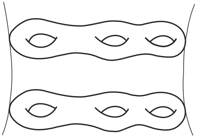

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$$\Phi(\sigma, w_1, w_2) = \begin{cases} \text{filling of } \mathbb{H}^3/\sigma_{w_1,w_2} & \text{if } (w_1, w_2) \neq (\infty, \infty) \\ \sigma & \text{if } (w_1, w_2) = (\infty, \infty) \end{cases}$$

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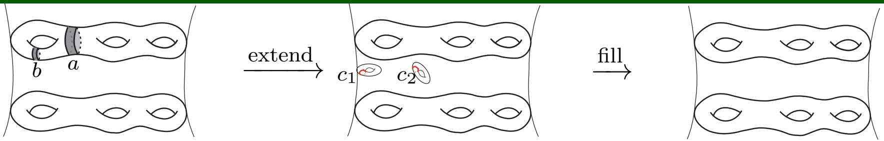

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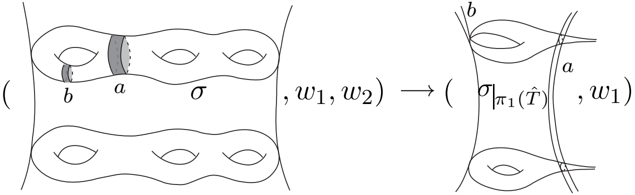
Theorem (Bromberg)

For any (σ, ∞, ∞), there is a neighborhood U in $\mathcal{A}_M$ such that $\Phi|_U : U \to \Phi(U) \subset AH(M)$ is a homeomorphism.

Define $\Pi : \mathcal{A}_M \to \mathcal{A}_{\hat{T}}$ by $(\sigma, w_1, w_2) \mapsto (\sigma|_{\pi_1(\hat{T})}, w_1)$



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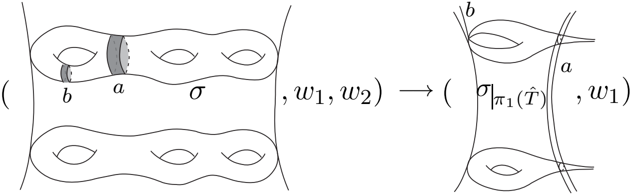

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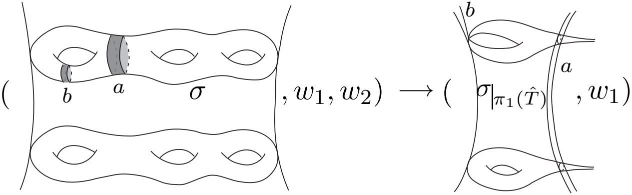

Lemma

There exists a point (σ_0, ∞), *a neighborhood* $U \subset \mathcal{A}_{\hat{T}}$, *subsets* $C_n \subset U$, *and some* $\delta > 0$ *such that for any* $(\sigma, w) \in C_n$ *and any* $(\sigma', w') \in U - C_n$

$$|w - w'| > \delta$$

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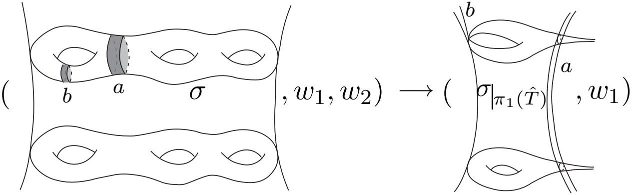
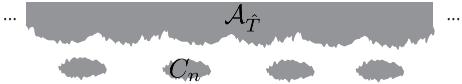


$\mathcal{A}_M \xrightarrow{\Pi}$

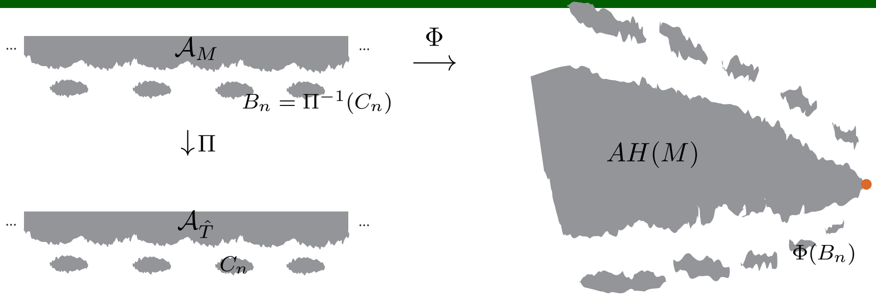


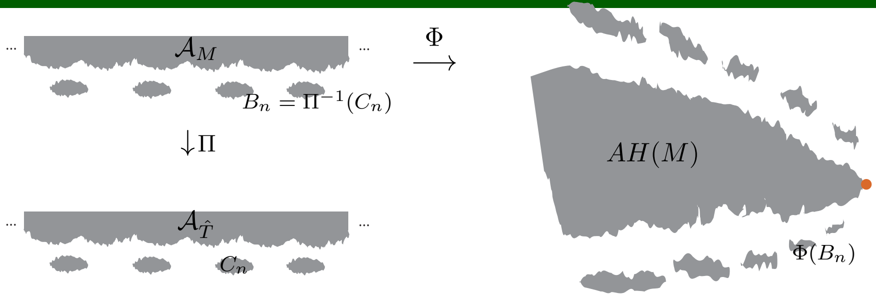
Lemma

There exists a point $(\sigma_0, \infty, \infty) \in U \subset \mathcal{A}_M$*, subsets* $\Pi^{-1}(C_n) \subset U$*, and some* $\delta > 0$ *such that for any* $(\sigma, w_1, w_2) \in \Pi^{-1}(C_n)$ *and* $(\sigma', w_1', w_2') \in U - \Pi^{-1}(C_n)$

$$|w_1 - w_1'| > \delta$$

$AH(M)$ is not locally connected





Filling Theorem implies complex length of γ in $\Phi(\sigma, w_1, w_2)$ is approximately

$$\ell + i\theta \approx \frac{4\pi Im(w_1)}{|w_1|^2} + i\frac{4\pi Re(w_1)}{|w_1|^2}$$

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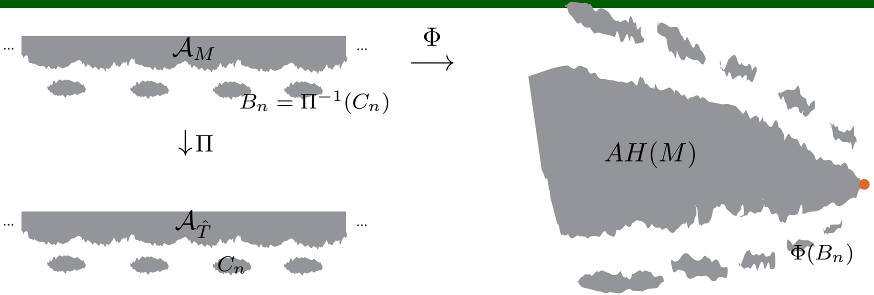

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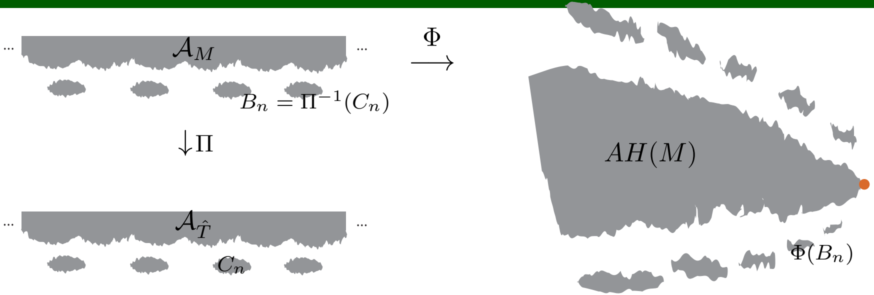
For all but finitely many n, $\overline{\Phi(B_n)}$ and $\overline{\Phi(U - B_n)}$ are disjoint



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Density $\Rightarrow$ $AH(M)$ is not locally connected.

- Replace punctured torus with four-punctured sphere

- Replace punctured torus with four-punctured sphere
- At which points is $AH(M)$ locally connected?

Thank you for listening!

Slides and preprints are available at:

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