Symplectic geometry of deformation spaces

Brice Loustau

July 15, 2010

Outline

- What is a symplectic structure?
- Goldman's construction
- Symplectic geometry of Teichmüller space $\mathcal{T}(S)$
- Symplectic geometry of $\mathcal{CP}(S)$

What is a symplectic structure?

Definition

A **symplectic structure** on a (smooth) manifold M is a **closed nondegenerate** 2-**form** ω.

Examples

Darboux coordinates

- If $(x_1, ..., x_n, y_1, ..., y_n)$ are global coordinates on M, $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ is a symplectic structure.
- (x_i, y_i) are called **Darboux coordinates** for the symplectic structure ω.

Examples

Darboux coordinates

- If $(x_1, \dots, x_n, y_1, \dots, y_n)$ are global coordinates on M, $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ is a symplectic structure.
- (x_i, y_i) are called **Darboux coordinates** for the symplectic structure ω.
- **Darboux theorem** : If (M, ω) is a symplectic manifold, there are Darboux coordinates in the neighborhood of any point.
- Any two symplectic manifolds of the same dimension are locally symplectomorphic.

Examples

The total space of a cotangent bundle

- Let $\pi : T^*M \to M$ be a cotangent bundle.
- Define the **canonical one-form** ξ on the total space T^*M by $\xi_\alpha = \pi^*\alpha$ [i.e. $\xi_\alpha(v) = \alpha(T_\alpha\pi(v))$ for $v \in T_\alpha T^*M$].

Examples

The total space of a cotangent bundle

- Let $\pi : T^*M \to M$ be a cotangent bundle.
- Define the **canonical one-form** ξ on the total space T^*M by $\xi_\alpha = \pi^*\alpha$ [i.e. $\xi_\alpha(v) = \alpha(T_\alpha\pi(v))$ for $v \in T_\alpha T^*M$].
- $\omega = -d\xi$ is the **canonical symplectic structure** of T^*M.

Examples

The total space of a cotangent bundle

- Let $\pi : T^*M \to M$ be a cotangent bundle.
- Define the **canonical one-form** ξ on the total space T^*M by $\xi_\alpha = \pi^*\alpha$ [i.e. $\xi_\alpha(v) = \alpha(T_\alpha\pi(v))$ for $v \in T_\alpha T^*M$].
- $\omega = -d\xi$ is the **canonical symplectic structure** of T^*M.
- In Hamiltonian mechanics, T^*M represents the phase space of a mechanical system.

Examples

Kähler manifolds

- Let M be a complex manifold. A **hermitian metric** h on M is a hermitian inner product in each tangent space.

Examples

Kähler manifolds

- Let M be a complex manifold. A **hermitian metric** h on M is a hermitian inner product in each tangent space.
- $g = \mathrm{Re}(h)$ is a Riemannian metric on M and $\omega = -\mathrm{Im}(h)$ is a nondegenerate 2-form.

Examples

Kähler manifolds

- Let M be a complex manifold. A **hermitian metric** h on M is a hermitian inner product in each tangent space.
- $g = \mathrm{Re}(h)$ is a Riemannian metric on M and $\omega = -\mathrm{Im}(h)$ is a nondegenerate 2-form.
- When ω is closed, it is a symplectic structure. h is called a **Kähler metric** and ω the **Kähler form** of h.

Examples

Complex symplectic structures

Definition

Let M be a complex manifold. A **complex symplectic structure** on M is a **complex-valued nondegenerate closed** $(2,0)$-**form** ω.

Examples

Complex symplectic structures

Definition

Let M be a complex manifold. A **complex symplectic structure** on M is a **complex-valued nondegenerate closed** $(2,0)$-**form** ω.

- Complex symplectic structures are holomorphic.

Examples

Complex symplectic structures

Definition

Let M be a complex manifold. A **complex symplectic structure** on M is a **complex-valued nondegenerate closed** $(2,0)$**-form** ω.

- Complex symplectic structures are holomorphic.
- If ω is a complex symplectic structure on M, then $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$ are real symplectic structures on M.

Goldman's construction

The character variety

- Let S be a closed oriented surface of genus $g \geqslant 2$, $\pi = \pi_1(S)$.
- Let G be a reductive Lie group, $\mathfrak{g}$ the Lie algebra of G.
- Let $\mathcal{X} = \mathcal{X}(S, G) = \mathrm{Hom}(\pi, G)//G$ be the character variety of S.

Note : I will ignore issues involving singularities.

Goldman's construction

The tangent space

At a "regular" point $[\rho]$ of $\mathcal{X}$, the tangent space is given by $T_{[\rho]}\mathcal{X} = H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho})$.

Explanation :

- let $\rho_t : \pi \to G$ be a path in $\mathrm{Hom}(\pi, G)$ such that $\rho_0 = \rho$.
- Define $\dot{\rho} : \alpha \in \pi \mapsto \frac{d}{dt}_{|t=0}\rho_t(\alpha)\rho^{-1}(\alpha)$.
- $\dot{\rho}$ is a map $\pi \to \mathfrak{g}$.
- $\rho_t(\alpha\beta) = \rho_t(\alpha)\rho_t(\beta) \Rightarrow \dot{\rho}(\alpha\beta) = \dot{\rho}(\alpha) + \mathrm{Ad}_{\rho(\alpha)}\dot{\rho}(\beta)$, i.e. $\dot{\rho} \in Z^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho})$.
- If $\rho_t = {g_t}^{-1}\rho g_t$ for some path $g_t \in G$, then $\dot{\rho}(\alpha) = \mathrm{Ad}_{\rho(\alpha)}\dot{g} - \dot{g}$, i.e. $\dot{\rho} \in B^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho})$.

Goldman's construction

The symplectic product in the tangent space

Let B be a nondegenerate symmetric bilinear form on $\mathfrak{g}$ invariant under the adjoint representation. If G is semisimple, take the Killing form (on $\mathfrak{sl}(n,\mathbb{R})$, $B(A,B) = 2n\mathrm{tr}(AB)$).

One can take the "cup product with B as coefficient pairing" :

$$\begin{array}{ccccc} H^1(\pi,\mathfrak{g}) \times H^1(\pi,\mathfrak{g}) & \to & H^2(\pi,\mathfrak{g}\otimes\mathfrak{g}) & \to & H^2(\pi,\mathbb{R}) \\ (u,v) & \mapsto & u \cup v & \mapsto & B(u \cup v) \end{array}$$

Note : the cup-product is the exterior (wedge) product in de Rham cohomology.

Goldman's construction

Note that $H^2(\pi, \mathbb{R}) \approx H^2(S, \mathbb{R}) \approx \mathbb{R}$ under integration.
We have defined a skew-symmetric nondegenerate bilinear form $\omega_{[\rho]} : (u, v) \mapsto \int B(u \cup v)$ on the tangent space $T_{[\rho]}\mathcal{X}$.

Goldman's construction

Note that $H^2(\pi, \mathbb{R}) \approx H^2(S, \mathbb{R}) \approx \mathbb{R}$ under integration.
We have defined a skew-symmetric nondegenerate bilinear form $\omega_{[\rho]} : (u, v) \mapsto \int B(u \cup v)$ on the tangent space $T_{[\rho]}\mathcal{X}$.

Theorem (Goldman, following Atiyah-Bott)

The 2-form ω_G on $\mathcal{X}$ defined by $(\omega_G)_{|T_{[\rho]}\mathcal{X}} = \omega_{[\rho]}$ is closed.

I will call this symplectic form **Goldman's symplectic form**.

Symplectic geometry of Teichmüller space

Goldman's symplectic form on Teichmüller space

Let $\mathcal{T}_{\mathrm{hyp}}(S)$ be the "hyperbolic" Teichmüller space of S (a point in $\mathcal{T}_{\mathrm{hyp}}(S)$ is a marked hyperbolic surface). It embeds as a connected component of $\mathcal{X}(S, G)$, where $G = PSL(2, \mathbb{R})$.

As a consequence of Goldman's construction, we get a symplectic structure ω_G on Teichmüller space.

Symplectic geometry of Teichmüller space

Teichmüller space as a complex manifold

Let $\mathcal{T}_{\mathbb{C}}(S)$ be the "conformal" Teichmüller space of S (a point X in $\mathcal{T}_{\mathbb{C}}(S)$ is a marked Riemann surface).

$\mathcal{T}_{\mathbb{C}}(S)$ is a complex manifold. The cotangent space is given by $T^*_X\mathcal{T}_{\mathbb{C}}(S) = Q(X)$, where $Q(X)$ is the space of holomorphic quadratic differentials on X.

Symplectic geometry of Teichmüller space

The Weil-Petersson metric

There is a Hermitian product on $Q(X)$ given by $(\phi, \psi) = \int_X \varphi\sigma^{-1}\bar{\psi}$, where σ is the area form of the Poincaré metric on X.
By duality, we get a Hermitian product on $T_X\mathcal{T}_{\mathbb{C}}(S)$. In other words, we have defined a Hermitian metric on $\mathcal{T}_{\mathbb{C}}(S)$, it is called the **Weil-Petersson metric**.

Symplectic geometry of Teichmüller space

The Weil-Petersson metric

There is a Hermitian product on $Q(X)$ given by $(\phi, \psi) = \int_X \varphi\sigma^{-1}\bar{\psi}$, where σ is the area form of the Poincaré metric on X.
By duality, we get a Hermitian product on $T_X\mathcal{T}_{\mathbb{C}}(S)$. In other words, we have defined a Hermitian metric on $\mathcal{T}_{\mathbb{C}}(S)$, it is called the **Weil-Petersson metric**.

Theorem (Ahlfors, Weil)

The Weil-Petersson metric is Kähler.

In other words, $\omega_{WP} = -\mathrm{Im}(h)$ is a (real) symplectic structure on $\mathcal{T}_{\mathbb{C}}(S)$.

A first identification

Goldman showed the two symplectic structures agree under the identification $\mathcal{T}_{\mathrm{hyp}}(S) \approx \mathcal{T}_{\mathbb{C}}(S)$ (given by uniformization) :

Theorem (Goldman)

$\omega_G = (-2)\omega_{WP}$

Symplectic geometry of Teichmüller space

The "Fenchel-Nielsen symplectic structure"

Let $(\alpha_1, \ldots, \alpha_N)$ be a pants decomposition of S and
$FN = (l_{\alpha_i}, \tau_{\alpha_i})$ be associated Fenchel-Nielsen coordinates on $\mathcal{T}(S)$.
$FN : \mathcal{T}(S) \to \mathbb{R}^{2N}$ are global coordinates on Teichmüller space.

$\omega_{FN} = \sum dl_i \wedge d\tau_i$ is a symplectic structure on Teichmüller space.
It depends *a priori* on the choice of the pants decomposition.

Symplectic geometry of Teichmüller space

Lemma (Wolpert)

If α, β are disjoint simple closed curves on S, then l_α and l_β Poisson-commute (for the Weil-Petersson symplectic structure) : $\{l_\alpha, l_\beta\} = 0$.

In particular, the functions l_{α_i} for the pants decomposition define a "completely integrable Hamiltonian system".

Symplectic geometry of Teichmüller space

Lemma (Wolpert)

If α, β are disjoint simple closed curves on S, then l_α and l_β Poisson-commute (for the Weil-Petersson symplectic structure) : $\{l_\alpha, l_\beta\} = 0$.

In particular, the functions l_{α_i} for the pants decomposition define a "completely integrable Hamiltonian system".

Theorem (Wolpert)

Fenchel-Nielsen coordinates are Darboux coordinates for the Weil-Petersson symplectic structure.

Conclusion

To summarize, we have constructed three examples of symplectic structures on Teichmüller space using apparently very different techniques, but they turn out to be the same :

Theorem

$\omega_G = \omega_{WP} = \omega_{FN}$

(maybe up to simple constants).

This both shows how natural this symplectic structure is and provides different angles to comprehend it.

Symplectic geometry of $\mathcal{CP}(S)$

Goldman's symplectic form on $\mathcal{CP}(S)$

Let $\mathcal{CP}(S)$ be the deformation space of marked projective structures on S. The holonomy map gives a local (biholomorphic) identification $\mathrm{hol} : \mathcal{CP}(S) \to \mathcal{X} = \mathcal{X}(S, G)$, where $G = PSL(2, \mathbb{C})$.

Symplectic geometry of $\mathcal{CP}(S)$

Goldman's symplectic form on $\mathcal{CP}(S)$

Let $\mathcal{CP}(S)$ be the deformation space of marked projective structures on S. The holonomy map gives a local (biholomorphic) identification $\mathrm{hol} : \mathcal{CP}(S) \to \mathcal{X} = \mathcal{X}(S, G)$, where $G = PSL(2, \mathbb{C})$.

As a consequence of Goldman's construction, we get a symplectic structure ω_G on $\mathcal{X}$. Because G is a complex Lie group, ω_G is a complex symplectic form (take the complex Killing form for B).

Pulling back by the holonomy map, we get a complex symplectic structure ω_G on $\mathcal{CP}(S)$.

Symplectic geometry of $\mathcal{CP}(S)$

Let $\sigma_{\mathcal{F}} : \mathcal{T}(S) \to \mathcal{CP}(S)$ be the (standard) Fuchsian section (given by uniformization) and $\mathcal{F}(S) = \sigma_{\mathcal{F}}(\mathcal{T}(S))$ be the standard Fuchsian slice in $\mathcal{CP}(S)$. It is a copy of Teichmüller space in $\mathcal{CP}(S)$.

Symplectic geometry of $\mathcal{CP}(S)$

Let $\sigma_{\mathcal{F}} : \mathcal{T}(S) \to \mathcal{CP}(S)$ be the (standard) Fuchsian section (given by uniformization) and $\mathcal{F}(S) = \sigma_{\mathcal{F}}(\mathcal{T}(S))$ be the standard Fuchsian slice in $\mathcal{CP}(S)$. It is a copy of Teichmüller space in $\mathcal{CP}(S)$.

The following diagram commutes :

$$\begin{tikzcd}
\mathcal{CP}(S) \arrow[r, "hol_{\mathcal{CP}}"] & \mathcal{X}(S, PSL(2,\mathbb{C})) \\
\mathcal{T}_{\mathrm{hyp}}(S) \arrow[u, "\sigma_{\mathcal{F}}"] \arrow[r, "hol_{\mathcal{T}}"] & \mathcal{X}(S, PSL(2,\mathbb{R})) \arrow[u, "\iota"']
\end{tikzcd}$$

Symplectic geometry of $\mathcal{CP}(S)$

Let $\sigma_{\mathcal{F}} : \mathcal{T}(S) \to \mathcal{CP}(S)$ be the (standard) Fuchsian section (given by uniformization) and $\mathcal{F}(S) = \sigma_{\mathcal{F}}(\mathcal{T}(S))$ be the standard Fuchsian slice in $\mathcal{CP}(S)$. It is a copy of Teichmüller space in $\mathcal{CP}(S)$.

The following diagram commutes :

$$\begin{tikzcd}
\mathcal{CP}(S) \arrow[r, "hol_{\mathcal{CP}}"] & \mathcal{X}(S, PSL(2,\mathbb{C})) \\
\mathcal{T}_{\mathrm{hyp}}(S) \arrow[u, "\sigma_{\mathcal{F}}"] \arrow[r, "hol_{\mathcal{T}}"] & \mathcal{X}(S, PSL(2,\mathbb{R})) \arrow[u, "\iota"']
\end{tikzcd}$$

As a consequence, Goldman's symplectic structure is the Weil-Petersson Kähler form in restriction to the Fuchsian slice : $\sigma_{\mathcal{F}}^* \omega_G = \omega_{WP}$.

Symplectic geometry of $\mathcal{CP}(S)$

Complex Fenchel-Nielsen coordinates on $\mathcal{QF}(S)$

- Let $\mathcal{QF}(S)$ be the (standard) quasifuchsian space of S. It is an open set in $\mathcal{T}(S)$ (which contains the Fuchsian slice).

Symplectic geometry of $\mathcal{CP}(S)$

Complex Fenchel-Nielsen coordinates on $\mathcal{QF}(S)$

- Let $\mathcal{QF}(S)$ be the (standard) quasifuchsian space of S. It is an open set in $\mathcal{T}(S)$ (which contains the Fuchsian slice).
- Kourouniotis and Tan introduced a system of global holomorphic coordinates on $\mathcal{QF}(S)$, called complex Fenchel-Nielsen coordinates.
- These coordinates $(l^{\mathbb{C}}{}_1, \ldots, l^{\mathbb{C}}{}_N, \tau^{\mathbb{C}}{}_1, \ldots \tau^{\mathbb{C}}{}_N)$ depend on the choice of a pants decomposition $(\alpha_1, \ldots, \alpha_N)$ of S, and can be thought of as a complexification of the real Fenchel-Nielsen coordinates on the Fuchsian slice.

Symplectic geometry of $\mathcal{CP}(S)$

Complex Darboux coordinates on $\mathcal{QF}(S)$

Theorem

Complex Fenchel-Nielsen coordinates are Darboux coordinates for the complex symplectic structure ω_G on $\mathcal{QF}(S)$

Symplectic geometry of $\mathcal{CP}(S)$

Complex Darboux coordinates on $\mathcal{QF}(S)$

Theorem

Complex Fenchel-Nielsen coordinates are Darboux coordinates for the complex symplectic structure ω_G on $\mathcal{QF}(S)$

Functions $l^{\mathbb{C}}{}_i$ define a holomorphic Hamiltonian flow called "complex twisting" (corresponds to -or is a generalization of- what authors might have called complex bending, complex earthquakes, quakebending...)

Symplectic geometry of $\mathcal{CP}(S)$

Another identification

- Let $T^*\mathcal{T}(S)$ be the total space of the holomorphic cotangent bundle of Teichmüller space. It is equipped with a canonical complex symplectic structure $\omega_{\mathbb{C}}$.

Symplectic geometry of $\mathcal{CP}(S)$

Another identification

- Let $T^*\mathcal{T}(S)$ be the total space of the holomorphic cotangent bundle of Teichmüller space. It is equipped with a canonical complex symplectic structure $\omega_{\mathbb{C}}$.
- Recall that the Schwarzian parametrization gives an identification $\mathcal{CP}(S) \to T^*\mathcal{T}(S)$.
- However, this identification depends on the choice of a "zero section" σ. Name $\tau_\sigma : \mathcal{CP}(S) \to T^*\mathcal{T}(S)$ the identification using the section σ.

Symplectic geometry of $\mathcal{CP}(S)$

Another identification

- Let $T^*\mathcal{T}(S)$ be the total space of the holomorphic cotangent bundle of Teichmüller space. It is equipped with a canonical complex symplectic structure $\omega_{\mathbb{C}}$.
- Recall that the Schwarzian parametrization gives an identification $\mathcal{CP}(S) \to T^*\mathcal{T}(S)$.
- However, this identification depends on the choice of a "zero section" σ. Name $\tau_\sigma : \mathcal{CP}(S) \to T^*\mathcal{T}(S)$ the identification using the section σ.
- How do the symplectic structures ω_G and $(\tau_\sigma)^*\omega_{\mathbb{C}}$ on $\mathcal{CP}(S)$ compare?

Another identification, continued

Theorem (Kawai)

If σ is any Bers slice, then $(\tau_\sigma)^*\omega_{\mathbb{C}} = i\omega_G$.

Another identification, continued

Theorem (Kawai)

If σ is any Bers slice, then $(\tau_\sigma)^*\omega_{\mathbb{C}} = i\omega_G$.

This result relates to the following facts :

- Bers slices are Lagrangian in $\mathcal{CP}(S)$.
- "Quasifuchsian reciprocity" (McMullen)
- If M is any 3-manifold, the restriction map : $\mathcal{X}(M, G) \to \mathcal{X}(\partial M, G)$ is a Lagrangian immersion (at "regular" points)

Conclusion and questions

Conclusion

As in the case of Teichmüller space, we have seen in various ways that $\mathcal{CP}(S)$ carries a natural (complex) symplectic structure.

Conclusion and questions

Conclusion

As in the case of Teichmüller space, we have seen in various ways that $\mathcal{CP}(S)$ carries a natural (complex) symplectic structure.

In fact, everything seems to be happening in $\mathcal{CP}(S)$ as a complexification of the situation in Teichmüller space.

Conclusion and questions

Conclusion

As in the case of Teichmüller space, we have seen in various ways that $\mathcal{CP}(S)$ carries a natural (complex) symplectic structure.

In fact, everything seems to be happening in $\mathcal{CP}(S)$ as a complexification of the situation in Teichmüller space.

The one situation that did not occur in the case of $\mathcal{CP}(S)$ was obtaining the symplectic structure as the imaginary part of some natural metric.

Conclusion and questions

Questions

- Does the complex symplectic structure on $\mathcal{CP}(S)$ come as the "imaginary part" of some hyperkähler metric on $\mathcal{CP}(S)$?

Conclusion and questions

Questions

- Does the complex symplectic structure on $\mathcal{CP}(S)$ come as the "imaginary part" of some hyperkähler metric on $\mathcal{CP}(S)$?
- How would such a hyperkähler metric compare to the canonical hyperkhäler metric on $T^*\mathcal{T}(S)$?

Conclusion and questions

Questions

- Does the complex symplectic structure on $\mathcal{CP}(S)$ come as the "imaginary part" of some hyperkähler metric on $\mathcal{CP}(S)$?
- How would such a hyperkähler metric compare to the canonical hyperkhäler metric on $T^*\mathcal{T}(S)$?
- What about the hyperkähler structure given by the description in terms of Higgs bundles?