Dynamics of actions of outer automorphism groups on $\mathrm{PSL}_2(\mathbb{C})$-character varieties

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Overview

The space $AH(M)$ of marked hyperbolic 3-manifold homotopy equivalent to a compact 3-manifold with boundary M sits inside the character variety

$$X(M) = Hom(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))//\mathrm{PSL}_2(\mathbf{C}).$$

We study the dynamics of the action of $\mathrm{Out}(\pi_1(M))$ on both $AH(M)$ and $X(M)$. The nature of the dynamics reflects the topology of M.

The quotient

$$AI(M) = AH(M)/\mathrm{Out}(\pi_1(M))$$

may naturally be thought of as the moduli space of unmarked hyperbolic 3-manifolds homotopy equivalent to M and its topology reflects the dynamics of the action.

2-dimensional warm-up

- Let F be a closed orientable surface of genus $g \geq 2$.
- $\mathcal{T}(F) = \{(\mathrm{S}, h) \mid \mathrm{S}\ \text{ a hyperbolic surface}, h : F \to \mathrm{S}\ \text{ o.p. homeo}\}/\sim$
- $(\mathrm{S}_1, h_1) \sim (\mathrm{S}_2, h_2)$ if there exists an o.p. homeo $j : \mathrm{S}_1 \to \mathrm{S}_2$ such that $j \circ h_1$ is homotopic to h_2.
- $\mathrm{S} = \mathbb{H}^2/\Gamma_{\mathrm{S}}$ where $\Gamma_{\mathrm{S}} \subset \mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}_+(\mathbb{H}^2)$ is discrete
- (S, h) gives rise to a discrete, faithful representation $h_* : \pi_1(F) \to \Gamma_{\mathrm{S}} \subset \mathrm{PSL}_2(\mathbb{R})$ which is well-defined up to conjugacy.
- So, $\mathcal{T}(F)$ embeds in

$$X_2(F) = Hom(\pi_1(F), \mathrm{PSL}_2(\mathbb{R}))//\mathrm{PSL}_2(\mathbb{R}))$$

- In fact, $\mathcal{T}(F)$ is an entire component of $X_2(F)$.

The action of $\mathrm{Mod}(F)$

- $\mathrm{Mod}(F)$ is the group of (isotopy classes of) o.p. homeos of F.
- $\mathrm{Mod}(F)$ is identified with an index two subgroup of $\mathrm{Out}(\pi_1(F))$.
- $\mathrm{Mod}(F)$ acts on $\mathcal{T}(F)$ and on $X_2(F)$.
- If $\phi \in \mathrm{Mod}(F)$, then ϕ takes $(\mathrm{S}, h) \in \mathcal{T}(F)$ to $(\mathrm{S}, h \circ \phi^{-1})$ and takes $\rho \in X_2(F)$ to $\rho \circ (\phi_*)^{-1}$.
- $\mathrm{Mod}(F)$ acts properly discontinuously, but not freely, on $\mathcal{T}(F)$ and its quotient is Moduli space

  $$\mathcal{M}(F) = \mathcal{T}(F)/\mathrm{Mod}(F).$$

  which is an orbifold.
- **Conjecture:** (Goldman) $\mathrm{Mod}(F)$ acts ergodically on the component of $X_2(F)$ not corresponding to $\mathcal{T}(F)$ or $\mathcal{T}(\bar{F})$.

Augmented Teichmüller space

- Augmented Teichmüller space $\widehat{\mathcal{T}}(F)$ is obtained by adding marked noded surfaces to $\mathcal{T}(F)$.
- Elements of $\widehat{\mathcal{T}}(F)$ have the form $(\gamma, \{\mathrm{S}_R\}_{R\in c(\gamma)}, h)$ where γ is a multicurve on F, $c(\gamma)$ is the collection of components of $F-\gamma$, $\cup\mathrm{S}_R$ is a finite area hyperbolic surface, and $h : F-\gamma \to \cup\mathrm{S}_R$ is an o.p. homeo.
- Equivalently, we may think of them as having the form $(\gamma, \{\rho_R\}_{R\in c(\gamma)})$ where each $\rho_R : \pi_1(R) \to \mathrm{PSL}_2(\mathbb{R})$ is a discrete faithful representation such that each element of $\rho_R(\pi_1(\partial R))$ is parabolic.

The Deligne-Mumford compactification

- The action of $\mathrm{Mod}(F)$ on $\mathcal{T}(F)$ extends continuously to an action on $\widehat{\mathcal{T}}(F)$.
- Its quotient
$$\widehat{\mathcal{M}}(F) = \widehat{\mathcal{T}}(F)/Mod(F)$$
is known as the Deligne-Mumford compactification of moduli space.

Kleinian surface groups

- $AH(F) = \{\ (N,h)\ \mid\ N\ \text{ hyperbolic }\ 3-\text{manifold}$ $h : F \rightarrow N\ \text{ homotopy }\ \text{equivalence}\}/\sim$
- $(N_1,h_1) \sim (N_2,h_2)$ if there exists an o.p. homeo $j : N_1 \rightarrow N_2$ such that $j \circ h_1$ is homotopic to h_2.
- $N = \mathbb{H}^3/\Gamma_N$ where $\Gamma_N \subset \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}_+(\mathbb{H}^3)$ is discrete
- (N,h) gives rise to a discrete, faithful representation $h_* : \pi_1(F) \rightarrow \Gamma_N \subset \mathrm{PSL}_2(\mathbb{C})$ which is well-defined up to conjugacy.
- So, $AH(F)$ embeds in

$$X(F) = Hom(\pi_1(F), \mathrm{PSL}_2(\mathbb{C}))//\mathrm{PSL}_2(\mathbb{C}))$$

- $AH(F)$ is a closed subset of $X(F)$, but is not an entire component.

Topology of $AH(F)$

- (Bers, Marden, Sullivan) The interior $QF(F)$ of $AH(F)$ is parameterized as $\mathcal{T}(F) \times \mathcal{T}(F)$.
- $\mathcal{T}(F)$ is identified with the diagonal in $QF(F)$.
- (Brock-C-Minsky) Elements of $AH(F)$ have been classified, but classification data is discontinuous, so does not yield a parameterization.
- (Brock-C-Minsky, Bromberg with Brock and Souto) $AH(F)$ is the closure of its interior $QF(F)$.
- (Bromberg,Magid) $AH(F)$ is not locally connected.

Action of $\mathrm{Mod}(F)$ on $AH(F)$

- $\mathrm{Mod}(F)$ acts on $AH(F)$ and on $X(F)$, again $\phi \in \mathrm{Mod}(F)$ takes $\rho \in X(F)$ to $\rho \circ (\phi_*)^{-1}$.
- (Thurston) If ϕ is pseudo-Anosov, then ϕ has a fixed point $\rho \in AH(F)$. N_ρ is the cover of the hyperbolic 3-manifold homeomorphic the mapping torus M_ϕ associated to the fibre subgroup.
- Therefore, $\mathrm{Mod}(F)$ does not act properly discontinuously on $AH(F)$ (or on $X(F)$.)
- Let $AI(F) = AH(F)/Mod(F)$ be the moduli space of hyperbolic 3-manifolds homotopy equivalent to F.
- (C-Storm) $AI(F)$ is not T_1, i.e. there are points which are not closed.

The conjectural picture

Conjecture: If $\mathcal{O} \subset X(F)$ is open, $\mathrm{Mod}(F)$-invariant and $\mathrm{Mod}(F)$ acts properly discontinuously on $\mathcal{O}$, then $\mathcal{O} \subset QF(F)$.

Remarks: (1) One can fairly easily show that $\mathcal{O}$ cannot intersect $\partial AH(F)$, using the fact that geometrically finite hyperbolic 3-manifolds with cusps are dense in $\partial AH(F)$.

(2) Work of Tan-Wong-Zhang and Cantat provides evidence for this conjecture in the case that F is a once punctured torus.

A Deligne-Mumford compactification of $AI(F)$

- One can form an augmented deformation space $\widehat{AH}(F)$ by adding points of the form $(\gamma, \{\rho_R\}_{R\in c(\gamma)})$ where again γ is a multicurve and each $\rho_R : \pi_1(R) \to \mathrm{PSL}_2(\mathbb{C})$ is a discrete faithful representation such that each element of $\rho_R(\pi_1(\partial R))$ is parabolic.
- The action of $\mathrm{Mod}(F)$ extends continuously to an action on $\widehat{AH}(F)$.
- (C-Storm) $\widehat{AI}(F) = \widehat{AH}(F)/\mathrm{Mod}(F)$ is sequentially compact.
- Notice that $\widehat{\mathcal{T}}(F) \subset \widehat{AH}(F)$, so $\widehat{\mathcal{M}}(F) \subset \widehat{AI}(F)$.

Deformation spaces of hyperbolic 3-manifolds

- Let M be a compact hyperbolizable 3-manifold with non-empty boundary.
- For simplicity, assume that ∂M contains no tori.
- $AH(M) = \{\ (N,h)\ \mid\ N \ \text{hyperbolic}\ \ 3-\text{manifold}$ $h: M \to N \ \text{homotopy}\ \ \text{equivalence}\}/\sim$
- $N = \mathbb{H}^3/\Gamma_N$ where $\Gamma_N \subset \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}_+(\mathbb{H}^3)$ is discrete
- (N,h) gives rise to a discrete, faithful representation $h_*: \pi_1(M) \to \Gamma_N \subset \mathrm{PSL}_2(\mathbb{C})$ which is well-defined up to conjugacy.
- So, $AH(M)$ embeds in

$$X(M) = Hom(\pi_1(M), \mathrm{PSL}_2(\mathbb{C}))//\mathrm{PSL}_2(\mathbb{C}))$$

- $AH(M)$ is a closed subset of $X(M)$, but is not an entire component.

The action of $\mathrm{Out}(\pi_1(M))$ on $X(M)$

- Let $\mathrm{Out}(\pi_1(M)) = \mathrm{Aut}(\pi_1(M))/\mathrm{Inn}(\pi_1(M))$.
- $\mathrm{Out}(\pi_1(M))$ acts on $X(M)$ preserving $AH(M)$ (as before, ϕ takes ρ to $\rho \circ \phi^{-1}$).
- It is an immediate consequence of the classical deformation theory of Kleinian groups that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on the interior $\mathrm{int}(AH(M))$ of $AH(M)$.
- Let $AI(M) = AH(M)/\mathrm{Out}(\pi_1(M))$ be the deformation space of unmarked hyperbolic 3-manifolds homotopy equivalent to M.
- (C-Storm) $AI(M)$ is T_1 if and only if M is not an untwisted interval bundle, i.e. $M \neq F \times [0,1]$ for a compact surface F.

Essential annuli in 3-manifold

- An **essential annulus** in M is an embedded annulus $A \subset M$ such that $\partial A \subset \partial M$, $\pi_1(A)$ injects into $\pi_1(M)$ and A cannot be homotoped (rel ∂A) into ∂M.
- An essential annulus is **primitive** if $\pi_1(A)$ is a maximal abelian subgroup of $\pi_1(M)$.
- M is **acylindrical** if it contains no essential annuli.
- (Johannson) $\mathrm{Out}(\pi_1(M))$ is finite if and only if M is acylindrical.

Domains of discontinuity for $\mathrm{Out}(\pi_1(M))$

- $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $X(M)$ if and only if M is acylindrical.
- **Question:** When does $\mathrm{Out}(\pi_1(M))$ act properly discontinuously on $AH(M)$?
- **Corollary:** (C-Storm) $\mathrm{Out}(\pi_1(M))$ act properly discontinuously on $AH(M)$ if and only if M contains no primitive essential annuli. Moreover, $AI(M)$ is Hausdorff if and only if M contains no primitive essential annuli.
- This corollary follows from the following results:

The main results

Theorem 1: (C-Storm) If M contains a primitive essential annulus, then $\mathrm{Out}(\pi_1(M))$ does not act properly discontinuously on $AH(M)$. Moreover, $AI(M)$ is not Hausdorff.

Theorem 2: (C-Storm) If M contains no primitive essential annuli, then $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on an open neighborhood $W(M)$ of $AH(M)$ in $X(M)$. Moreover, $AI(M)$ is Hausdorff.

A generalization of Theorem 2

Theorem 3: If M has incompressible boundary and is not an interval bundle, then there exists an open subset $W(M)$ of $X(M)$ such that

1. $W(M)$ is invariant under $\mathrm{Out}(\pi_1(M))$,
2. $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$,
3. $\mathrm{int}(AH(M)) \subset W(M)$, and
4. $W(M) \cap \partial AH(M)$ is non-empty.

Remark: In the case that M is a handlebody, Yair Minsky showed that the set of primitive stable representations is a subset of $AH(M)$ with all these properties.

Twisted interval bundles

Theorem 3: (Michelle Lee) If M is a twisted interval bundle, then there exists an open subset $W(M)$ of $X(M)$ such that

1. W is invariant under $\mathrm{Out}(\pi_1(M))$,
2. $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$,
3. $\mathrm{int}(AH(M)) \subset W(M)$, and
4. $W(M) \cap \partial AH(M)$ is non-empty.

Remark: Lee's technique of proof is necessarily quite different than that of Canary-Storm. Her work is inspired by work of Minsky.

A corollary of Lee's work

Corollary: (Michelle Lee) If M has incompressible boundary, then there exists an open subset W of $X(M)$ such that

1. W is invariant under $\mathrm{Out}(\pi_1(M))$,
2. $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on W,
3. $\mathrm{int}(AH(M)) \subset W$, and
4. $W \cap \partial AH(M)$ is non-empty.

if and only if M is not an untwisted interval bundle.