Harmonic Deformations of Hyperbolic Structures

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Outline

1. Background: deformations of geometric structures, infinitesimal deformations, cohomology
2. Harmonic representatives, infinitesimal rigidity
3. Application: Hyperbolic Dehn surgery
4. Other applications/extensions

Most of this talk will describe joint work with Steve Kerckhoff.

Geometric structures

(Following Klein, Ehresmann, Thurston)

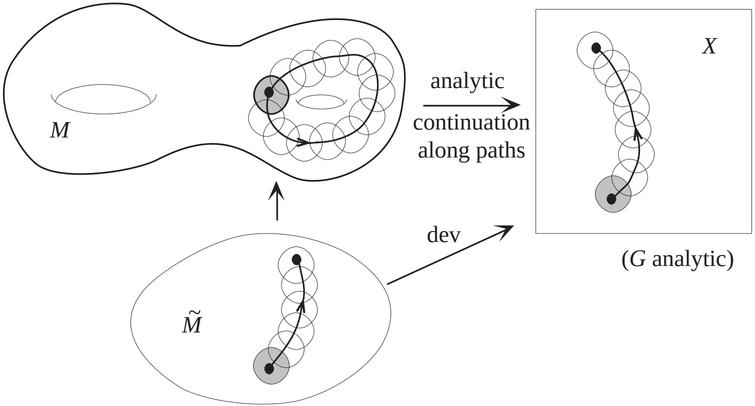
A *geometry* is a pair (G, X) where G is a Lie group acting transitively and analytically on a manifold X.

- *hyperbolic geometry* arises when $G = \mathrm{Isom}(\mathbb{H}^n)$ is the group of isometries of hyperbolic space $X = \mathbb{H}^n$.

A *(G, X)-structure* on a manifold M is given by a collection of coordinate charts $\phi_i : U_i \to X$ covering M with transition functions $\phi_i \circ \phi_j^{-1}$ given by restrictions of elements of G.

Developing Map and Holonomy Representation

Analytic continuation of coordinate charts gives a *developing map* $\Phi : \tilde{M} \to X$, where $\tilde{M}$ is the universal cover of M.



This is a local diffeomorphism satisfying the equivariance condition

$$\Phi(\gamma \cdot m) = \rho(\gamma)\Phi(m), \text{ for all } m \in \tilde{M}, \gamma \in \pi_1(M)$$

where $\pi_1(M)$ acts on $\tilde{M}$ by covering transformations and

$$\rho : \pi_1(M) \to G$$

is a homomorphism called the *holonomy representation* for the structure.

We obtain an *equivalent* geometric structure if Φ is replaced by $k \circ \Phi \circ \tilde{f}$ where $k \in G$ and $\tilde{f}$ is the lift of a diffeomorphism $f : M \to M$ isotopic to the identity.

Basic fact (Ehresmann, Thurston)
Nearby geometric structures (up to equivalence) correspond to nearby representations (up to conjugacy).

Deformations

A *1-parameter family* of (G, X)-structures on a manifold M is given by a smooth family of developing maps

$$\Phi_t : \tilde{M} \to X, \quad t \in \mathbb{R},$$

and associated family of holonomies

$$\rho_t : \pi_1(M) \to G.$$

If X *is a Riemannian manifold and* $G = \mathrm{Isom}(X)$, *then the metric on* X *pulls back via* Φ_t *to give a family of metrics* g_t *on* M.

We write $\Phi = \Phi_0$, $\rho = \rho_0$ *and* $g = g_0$.

The tangent vector to such a deformation gives an infinitesimal deformation of the geometric structure. There are several useful ways to think about this.

Variation of holonomy

The derivative of $\rho_t(\gamma)$ at $t = 0$, for $\gamma \in \pi_1(M)$, gives

$$\dot{\rho} : \pi_1(M) \to \mathcal{G}$$

where $\mathcal{G}$ is the Lie algebra of G. This satisfies the cocycle condition

$$\dot{\rho}(\gamma_1\gamma_2) = \dot{\rho}(\gamma_1) + Ad\rho(\gamma_1)\dot{\rho}(\gamma_2),$$

so $\dot{\rho}$ is a *1-cocycle in group cohomology* $Z^1(\pi_1 M; Ad\rho)$.

For *trivial deformations* $\rho_t = k_t \rho k_t^{-1}$, where $k_t \in G$, $\dot{\rho}$ is a *coboundary* in $B^1(\pi_1 M; Ad\rho)$.

So we obtain a well-defined cohomology class

$$[\dot{\rho}] \in H^1(\pi_1(M); Ad\rho).$$

Representation Spaces and Cohomology

This leads to the following observations of A. Weil:

- $Z^1(\pi_1 M; Ad\rho)$ is the Zariski tangent space to the space of representations $Hom(\pi_1 M, G)$ at ρ.
- $B^1(\pi_1 M; Ad\rho)$ is the Zariski tangent space to the orbit of ρ under conjugation by G.
- $H^1(\pi_1 M; Ad\rho)$ can be regarded as the Zariski tangent space to the character variety $Hom(\pi_1 M, G)/G$ of representations up to conjugation.

- **Theorem** (Weil) If $H^1(\pi_1 M; Ad\rho) = 0$ then ρ is locally rigid, i.e. all nearby representations are conjugate to ρ.

  Thus cohomology vanishing theorems give rise to local rigidity theorems.

Variation of metrics

For $x \in M$, $t \in \mathbb{R}$ we have an inner product

$$g_t : T_x M \times T_x M \to \mathbb{R}.$$

Differentiating at $t = 0$ gives

$$\dot{g} : T_x M \times T_x M \to \mathbb{R}.$$

Equivalently, this can be written

$$\dot{g}(v, w) = g(\eta v, w)$$

where $\eta \in \Omega^1(M; TM)$ is a 1-form with values in TM
(with $\eta : T_x M \to T_x M$ symmetric at each point).

Trivial deformations $g_t = \phi_t^*(g)$ arise if we change the metric by diffeomorphisms $\phi_t : M \to M$. (So each g_t is isometric to g.)

De Rham cohomology

There is an isomorphism between group cohomology and *de Rham cohomology*

$$H^1(M; E) \cong H^1(\pi_1(M); Ad\rho)$$

where E is the flat $\mathcal{G}$-bundle over M

$$E = (\tilde{M} \times \mathcal{G})/\sim$$

where $(x, v) \sim (\gamma x, Ad\rho(\gamma)v)$ for $\gamma \in \pi_1(M)$.
For a hyperbolic structure, E is the bundle of (germs of) infinitesimal hyperbolic isometries on M.

In fact, we can obtain a closed 1-form $\omega \in \Omega^1(M; E)$ directly from the *variation of developing maps*, but I won't describe this today.

How are these approaches related?

- Integration of ω around loops in M gives the variation in holonomy $\dot{\rho}$.
- The variation in metric is also determined by ω:
  For hyperbolic geometry, the Lie algebra $\mathcal{G}$ splits at each point of $X = \mathbb{H}^n$ into a direct sum

$$\textit{infinitesimal translations} \oplus \textit{infinitesimal rotations}.$$

  The *translational part* of ω gives a *TM*-valued 1-form; the *symmetric part* of this is the form η describing the variation in the metric.

Harmonic representatives

From now on we assume that $X = \mathbb{H}^n$, $G = \mathrm{Isom}(\mathbb{H}^n)$, and E is the bundle of infinitesimal isometries of $\mathbb{H}^n$.

If we choose a natural metric on the fibres of E, then Hodge theory gives a *harmonic representative* ω (closed and co-closed) for each cohomology class in $H^1(M; E)$.

Existence and uniqueness of such a form is a standard fact on closed Riemannian manifolds. The harmonic representative minimizes the L^2 norm.

For non-compact manifolds or manifolds with boundary, additional asymptotic or boundary conditions are needed for uniqueness.

We can now use geometry to help compute cohomology.

Infinitesimal harmonic deformations

Let $\eta \in \Omega^1(M; TM)$ be the translational part of the harmonic form $\omega \in \Omega^1(M; E)$. This is L^2-*orthogonal* to the trivial infinitesimal deformations of the metric.

Let D denote the *exterior covariant derivative*, $D : \Omega^k(M; TM) \to \Omega^{k+1}(M; TM)$ and D^* is its formal adjoint.

Then η harmonic implies

$$D^*\eta = 0.$$

For *hyperbolic* n-manifolds (constant curvature $K = -1$) we also have a *Weitzenböck formula* (for harmonic η):

$$D^*D\eta = -(n-2)\eta.$$

Now we can use a *Bochner* type argument:

$$\int_M \langle D^* D\eta, \eta\rangle = -(n-2)\int_M \langle \eta, \eta\rangle.$$

Integrating by parts gives

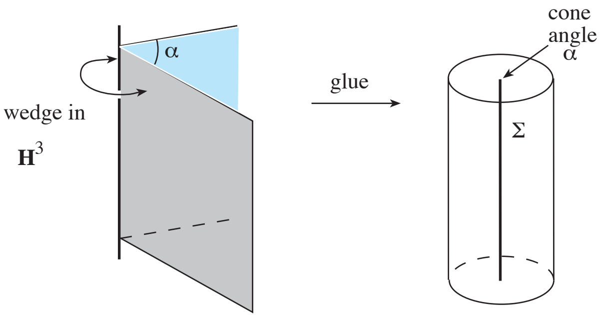
$$\int_M \langle D\eta, D\eta\rangle + (n-2)\int_M \langle \eta, \eta\rangle = \text{ boundary term.}$$

If M is closed, then boundary term $= 0$. So if $n \geq 3$, $\eta = 0$.

This proves an **Infinitesimal Rigidity Theorem:**
there are no infinitesimal deformations of the hyperbolic structure on a closed hyperbolic manifold of dimension $n \geq 3$.
(This result of Calabi and Weil gives a local version of the *Mostow rigidity theorem*.)

Extension to hyperbolic cone manifolds

Let N be a 3-dimensional hyperbolic *cone manifold* with cone angles $\leq 2\pi$ along a knot or link Σ.
Local picture near Σ:



Then we can choose harmonic representatives with controlled behaviour near the singular set Σ. (This requires some care as $M = N - \Sigma$ is non-compact and the metric is incomplete.)

We obtain a harmonic form $\omega = \omega_0 + \omega_c \in \Omega^1(M; E)$ where ω_0 is a *standard form* changing the holonomy near Σ, and ω_c is in L^2.

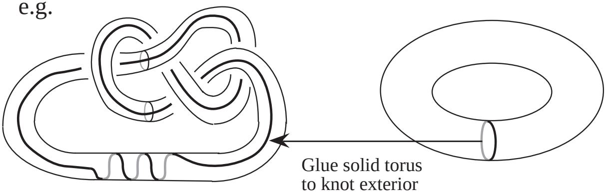
Then we remove a tube around Σ, and show that the boundary term $\to 0$ as tube radius $\to 0$ provided **cone angles are fixed** and $\leq 2\pi$. This proves *infinitesimal rigidity rel cone angles*.

Surprisingly, these methods can also used to understand situations where non-trivial deformations exist. In this case the arguments can be used to estimate the change in geometry as the hyperbolic structure is deformed.

Next: use this to study hyperbolic Dehn filling on cusped hyperbolic 3-manifolds.

Dehn Filling

e.g.



Let M = interior of a 3-manifold $\bar{M}$ with $\partial\bar{M} = T$= torus.
If γ = simple closed curve on $\partial\bar{M}$, we form

$$M(\gamma) = \gamma\text{-Dehn filling on } M,$$

by attaching a solid torus W to $\bar{M}$, gluing the boundaries together by a homeomorphism so γ bounds a disk in W.

- Choices of γ (up to isotopy)
  $\leftrightarrow$ relatively prime $(p, q) \in \mathbb{Z}^2 = \pi_1(T)$.

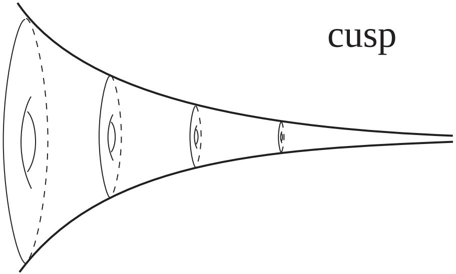
Theorem (Wallace, Lickorish)
Every closed orientable 3-manifold can be obtained by Dehn filling on a link complement $M = S^3 - \text{link}$.

Thurston showed "most" knot and link complements are *hyperbolic*. So we want to understand Dehn filling on *cusped* hyperbolic 3-manifolds: complete, non-compact, finite volume.

Hyperbolic Dehn Filling Theorem (Thurston)
Let M be a cusped hyperbolic 3-manifold. Then "almost all" Dehn fillings on M are *hyperbolic*, i.e. if we exclude finitely many fillings for each cusp, all others are hyperbolic.

We would like to make this more precise.

Theorem 1 (H–K)
Let M be a 1-cusped hyperbolic 3-manifold,
T a horospherical (Euclidean) torus cusp cross-section,
$L(\gamma)$ the length of the Euclidean geodesic on T homotopic to the surgery curve γ, and
$\hat{L}(\gamma) = \dfrac{L(\gamma)}{\sqrt{\text{Area}(T)}}$ the normalised geodesic length of γ.
Then if $\hat{L}(\gamma) > 7.515$, $M(\gamma)$ is hyperbolic.



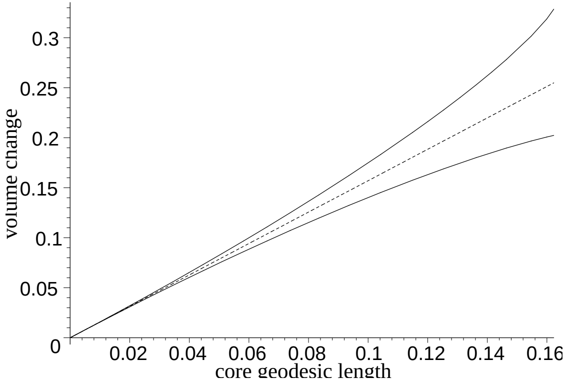
Continuous Dehn Drilling

We can also *drill out* short closed geodesics in hyperbolic 3-manifolds, e.g.

Theorem 2 (H–K)
Let M be a closed hyperbolic 3-manifold and τ a shortest closed geodesic in M.
If length$(\tau) \leq 0.162$, then the hyperbolic structure on M can be deformed to the (complete) hyperbolic structure on $M - \tau$.

We also get *good estimates on the changes in geometry*.

e.g. volume change $\Delta V = \mathrm{Vol}(M - \tau) - \mathrm{Vol}(M)$.



The dotted line shows the asymptotic formula of Neumann-Zagier: $\Delta V \sim \frac{\pi}{2}\ell$ as $\ell = \mathrm{length}(\tau) \to 0$.

Sketch of Proof of Theorem 1

Idea: Deform the complete structure on M to obtain hyperbolic structures on $M(\gamma)$ with **cone type singularities** along $\Sigma =$ core circle of added solid torus.

Aim: Increase the cone angle α along Σ

from $\alpha = 0$ (complete case)

to $\alpha = 2\pi$ (desired hyperbolic structure on $M(\gamma)$).

In general, this can't be done — degeneration may occur. However we show this **is** possible if $\hat{L}(\gamma)$ is sufficiently long.

Step 1. (Thurston '78)

Can always increase α from 0 to some $\epsilon > 0$. (By proof of the hyperbolic Dehn surgery theorem.)

Step 2. (H-K, J. Diff. Geom. '98)
Can always increase α slightly if $\alpha \leq 2\pi$.
Further, nearby hyperbolic cone manifolds are parametrized uniquely by their cone angles.

This follows from an analysis of the space of representations $\pi_1(M) \to PSL_2(\mathbb{C})$ and the previous **Rigidity Theorem**:

for a hyperbolic cone manifold with cone angles $\leq 2\pi$, there are no deformations **keeping cone angles fixed**.

Why can we change the cone angle?

Let $\mathcal{R} = R(\pi_1(M), PSL_2(\mathbb{C}))$ denote the space of all representations $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ up to conjugacy; this is an algebraic variety (*character variety*).

Given a hyperbolic cone manifold structure on $(M(\gamma), \Sigma)$ we have a holonomy representation $\rho_0 : \pi_1(M) \to PSL_2(\mathbb{C})$ such that $\rho_0(\gamma)$ is *elliptic* (rotation by the cone angle).

In general, nearby hyperbolic cone structures on $M(\gamma)$ correspond to $\rho \in \mathcal{R}$ near ρ_0 such that $\rho(\gamma)$ is elliptic.

A clever dimension counting argument (due to Thurston) shows that near ρ_0, $\dim_{\mathbb{C}} \mathcal{R} \geq$ number of cusps of M.

In particular, deformations of the representation ρ_0 exist. But there are no deformation *fixing* the cone angles, so there must be deformations *changing* the cone angle!

This idea plus an implicit function theorem argument gives the local parametrization result.

Step 3. (H-K, Annals of Math. 2005)
No degeneration occurs as the cone angle α increases to 2π, provided $\hat{L}(\gamma)$ is large enough.

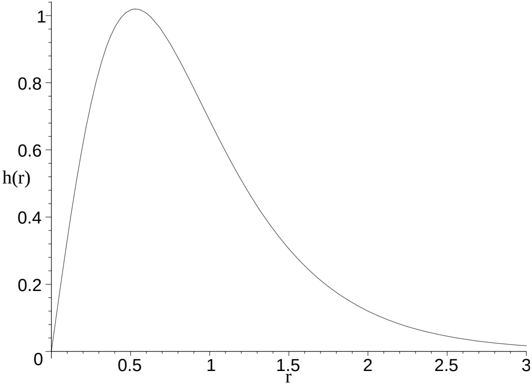
This uses:
(i) An **effective** (quantitative) version of **rigidity**:
Estimate the change in geometry near Σ, giving control on change in core geodesic length ℓ as α varies.

(This involves analyzing the boundary term in the previous Bochner argument more closely — there is a *standard model* changing cone angles plus correction terms).

(ii) **A tube packing argument** implies

$$\alpha\ell \geq h(R) = 3.3957\frac{\tanh(R)}{\cosh(2R)},$$

where $R =$ tube radius (with max when $R \approx 0.531$).



Initially:

$\alpha = 0$, $\ell = 0$, $R = \infty > 0.531$ (to the right of the "hump").

So: if $\alpha\ell < 1.019... = \max(h)$ throughout a deformation, then $R > 0.531$ throughout the deformation.

Combining this with (i) gives:

(iii) **Control on tube radius**

If $\hat{L}(\gamma) > 7.515$, then $R \geq R_0 = 0.531$ throughout any deformation with $0 \leq \alpha \leq 2\pi$.

(iv) **Control on Volume**

By Schläfli's formula, the change in volume during the deformation satisfies:

$$\frac{dV}{d\alpha} = -\frac{1}{2}\ell,$$

where ℓ denotes the length of the singular locus.

Hence the volume *decreases* as the cone angles increase, so is always bounded above by $Vol(M)$.

In fact, Schläfli's formula together with the estimates for ℓ from (i) also give us good upper and lower bounds on the change in volume.

(v) A **geometric limit** argument then shows:

$R \geq R_0$, $Vol \leq Vol(M)$ implies

no degeneration can occur for $0 \leq \alpha \leq 2\pi$.

Conclusion:

$M(\gamma)$ has a non-singular hyperbolic structure if $\hat{L}(\gamma) > 7.515$. This proves Theorem 1!

Some extensions and recent developments

• Results extend to *arbitrary* cone angles, if there is a *large tube* around Σ (radius $> \operatorname{arctanh}(\frac{1}{\sqrt{3}}) \approx 0.6585$). This involves finding harmonic representatives on a compact manifold, satisfying good boundary conditions.

• This gives general estimates on the size and shape of Thurston's "Hyperbolic Dehn surgery space" of a cusped hyperbolic manifold, and good estimates on the variation of geometry as the complete structure is deformed.

• Bromberg has extended this harmonic deformation theory to geometrically finite hyperbolic cone manifolds. There are many applications to Kleinian groups by Bromberg and his collaborators (e.g. the proof of Bers Density Conjecture involves deforming cone angles from 4π to 2π).

• Can estimate the geometry of link complements (e.g. volumes, cusp shapes) in terms of the combinatorics of link projections (Purcell).

• There are (weaker) results for deformations of Einstein metrics in higher dimensions (Montcouquiol).

• Kerckhoff-Storm have given analogues of hyperbolic Dehn surgery on certain infinite volume hyperbolic 4-orbifolds. They also have rigidity results for hyperbolic structures on compact manifolds (with totally geodesic boundary) of dimension ≥ 4.

• Weiss, Montcouquiol-Mazzeo have proved local rigidity rel cone angles for hyperbolic cone-manifolds with *an arbitrary graph as singular locus*, provided all cone angles are $\leq 2\pi$.

Further, deformations preserving the combinatorial type of singular locus are locally parametrized by the cone angles.

• Application: geometry of convex polyhedra in $\mathbb{H}^3$:

Stoker Conjecture (1968): Is a convex polyhedron in a space of constant curvature determined by its dihedral angles?

In euclidean space need to modify this question: Do the dihedral angles determine the face angles?

In the spherical case, Jean-Marc Schlenker found counterexamples.

Doubling the convex polyhedron gives a cone manifold with underlying space S^3, cone angles $\leq 2\pi$.

Then the results of Weiss, Montcouquiol, Mazzeo show:

• Convex polyhedra in $\mathbb{H}^3$ are locally determined by their dihedral angles. This work also proves a local version of the Euclidean Stoker Conjecture.

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