On the topology of $\mathcal{H}(2)$

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Translation surface

Definition

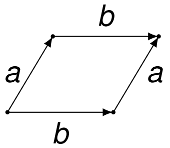
Translation surface is a flat surface with conical singularities such that the holonomy of any closed curve (which does not pass through the singularities) is a translation of $\mathbb{R}^2$.

Basic properties

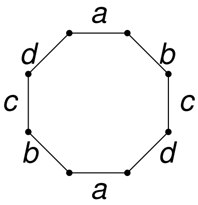
- Cone angles at singularities must belong to $2\pi\mathbb{N}$,
- A tangent vector at a regular point can be extended to a parallel vector field,
- Correspondence between a translation surface with a unitary parallel vector field and a holomorphic 1-form on a Riemann surface, zero of order $k \longleftrightarrow$ singularity with cone angle $(k+1)2\pi$.

Examples

- Flat tori (without singularities)



- Surfaces obtained from polygons by identifying sides which are parallel, and have the same length



Notations and terminologies

$\mathcal{H}(2)$ is the moduli space of pairs (M, ω) where M is a Riemann surface of genus 2 and ω is a holomorphic 1-form on M having only one zero which is of order 2. Equivalently, $\mathcal{H}(2)$ is the moduli space of translation surfaces of genus 2 having only one singularity with cone angle 6π.

Remark

- The unique zero of ω must be a Weierstrass point of M.
- Every Riemann surface of gennus 2 is hyper-elliptic, and therefore has exactly 6 Weierstrass points.

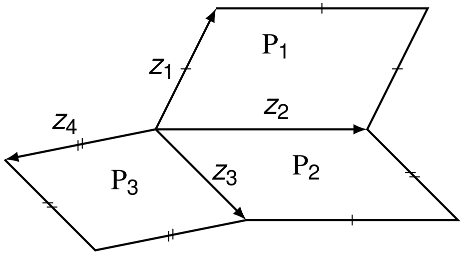
Notations and terminologies

We denote by $\mathcal{M}(2)$ the quotient $\mathcal{H}(2)/\mathbb{C}^*$ which is the set of pairs (M, W), where M is a Riemmann surface of genus 2, and W is a marked Weierstrass point of M.

A saddle connection on a translation surface is a geodesic segment joining two singularities, which may coincide. In the case of $\mathcal{H}(2)$, every saddle connection is a geodesic joining the unique singularity to itself.

Construction from parallelograms

We represent a parallelogram in $\mathbb{R}^2$ up to translation by a pair of complex numbers (z_1, z_2) such that $\mathrm{Im}(z_1\bar{z}_2) > 0$.
Given three parallelograms $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$ represented by the pairs $(z_1, z_2), (z_2, z_3)$, and (z_3, z_4) respectively, we can construct a surface in $\mathcal{H}(2)$ by the following gluing:



Construction from parallelograms

Proposition

Every surface in $\mathcal{H}(2)$ can be obtained from the previous construction.

Consequently, on every surface in $\mathcal{H}(2)$, there always exist a family of 6 saddle connections which decompose the surface into 3 parallelograms. We will call such families parallelogram decompositions of the surface.

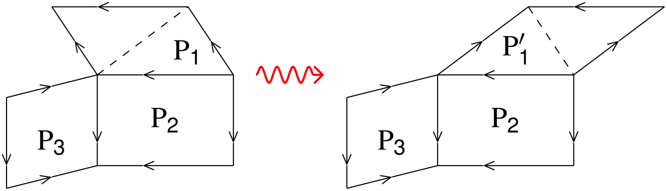
Construction from parallelograms

Question

- Which triples of parallelograms give the same surface in $\mathcal{H}(2)$?
- Given a surface in $\mathcal{H}(2)$, describe the set of parallelogram decompositions of this surface.

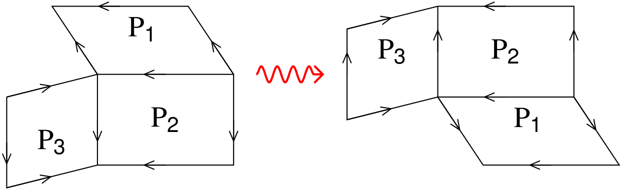
Elementary moves

T-move: changing P_1



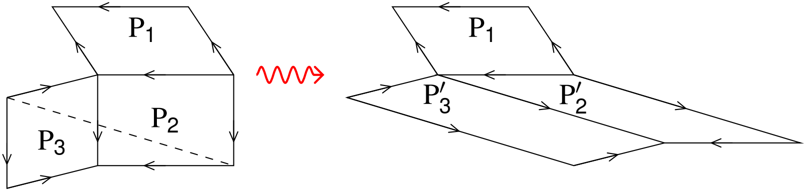
Elementary moves

S-move: permuting (P_1, P_2, P_3)



Elementary moves

R-move: changing P_2 and P_3



Elementary moves

Theorem

Two triples of parallelograms give rise to the same surface in $\mathcal{H}(2)$ if and only if one can be transformed to the other by a sequence of elementary moves.

Parallelogram decompositions

Given a surface Σ in $\mathcal{H}(2)$, we have elementary moves corresponding to T, S, R in the set of parallelogram decompositions. Those moves can be realized by homeomorphisms of the surface.

One can associate to each parallelogram decomposition of Σ a unique canonical basis of of $H_1(\Sigma, \mathbb{Z})$, then the actions of the corresponding homeomorphisms on $H_1(\Sigma, \mathbb{Z})$ in this basis given by the matrices T, S, and R.

Let Γ denote the group generated by T, S and R.

The matrices T, S, R

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$S = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix};$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Properties of Γ

- T and R commute, $S^2 = -\mathrm{Id}$,
- $\Gamma \subsetneq \mathrm{Sp}(4,\mathbb{Z})$, the action of Γ on $(\mathbb{Z}/2\mathbb{Z})^4 \setminus \{0\}$ has two orbits, but the action of $\mathrm{Sp}(4,\mathbb{Z})$ is transitive.
- Γ is not normal in $\mathrm{Sp}(4,\mathbb{Z})$.
- Γ contains $\mathrm{SL}(2,\mathbb{Z})$ as a proper subgroup.

Jacobian locus

For $g \geqslant 1$, the Siegel upper half space $\mathfrak{H}_g$ is the set of $g \times g$ complex symmetric matrices whose imaginary part is positive definite.

The Jacobian locus $\mathfrak{J}_g$ is the subset of $\mathfrak{H}_g$ consisting of period matrices associated to canonical homology bases of Riemann surfaces of genus g.

The moduli space $\mathfrak{M}_g$ of Riemann surfaces of genus g can be identified with $\mathfrak{J}_g/\mathrm{Sp}(2g, \mathbb{Z})$.

Jacobian locus

- Case $g = 1 : \mathfrak{J}_1 = \mathfrak{H}_1 = \mathbb{H}$ the hyperbolic upper half plan.
- Case $g = 2 : \mathfrak{J}_2 \subsetneq \mathfrak{H}_2$, the complement is a countable union of copies of $\mathfrak{H}_1 \times \mathfrak{H}_1$.

Main result

Theorem

The space $\mathcal{M}(2)$, that is the set of pairs (Riemann surface of genus 2, distinguished Weierstrass point), can be identified with the quotient $\mathfrak{J}_2/\Gamma$.

Main result

Main ideas:

- Generalizing the notion of "parallelogram decomposition" by taking into account the action of the hyperelliptic involution on $\pi_1(M, W)$.
- A connectivity result on a subset of the set of simple closed curves on M.
- Hyperellipticity of Riemann surfaces of genus 2, and Θ function.

Corollary

Corollary

We have $[\mathrm{Sp}(4,\mathbb{Z}) : \Gamma] = 6$*.*

Idea: there exists a map $\rho : \mathcal{M}(2) \longrightarrow \mathfrak{M}_2$ which is *generically* six to one.

Remark

Let $\mathrm{Mod}_{0,6}$ denote the mapping class group of the sphere with 6 punctures. The fundamental group of $\mathcal{M}(2)$ is the subgroup of $\mathrm{Mod}_{0,6}$ fixing a distinguished puncture.

The universal cover map factors through $\mathfrak{J}_2$, therefore we have a surjective homomorphism from $\pi_1(\mathcal{M}(2))$ onto Γ (more precisely $\Gamma/\{\pm\mathrm{Id}\}$).