Surfaces with large cone angles

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Motivation: Do,Norbury

Weil-Petersson volumes and cone surfaces, (2005)

- Mapping class group $\mathcal{MCG}$.
- Teichmuller space $= \mathcal{T}(\Sigma)$, ω_{WP} – $\mathcal{MCG}$-invar. Kahler form.
- Moduli space $= \mathcal{T}(\Sigma)/\mathcal{MCG}$, $(\omega_{WP})^{\frac{-3\chi(\Sigma)}{2}}$– symplectic vol.

Symplectic volume of the moduli space of a surface

- with points (= number).
  Wolpert (1982), Penner, Harer-Zagier
- with boundary (=polynomial).
  Nakanishi-Naatanen (2001), Mirzakhani(2003).

$$\begin{aligned} \text{torus, one hole,}\ \ V_1(l_1) &= \frac{1}{24}(4\pi^2 + l_1^2) \\ \text{torus, two hole,}\ \ V_1(l_1, l_2) &= \frac{1}{192}(4\pi^2 + l_1^2 + l_2^2)(12\pi^2 + l_1^2 + l_2^2) \end{aligned}$$

Motivation: Do,Norbury

Weil-Petersson volumes and cone surfaces, (2005)

$$
\begin{aligned}
V_1(l_1) &= \frac{1}{24}(4\pi^2 + l_1^2) \\
V_1(l_1, l_2) &= \frac{1}{192}(4\pi^2 + l_1^2 + l_2^2)(12\pi^2 + l_1^2 + l_2^2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dl_2} V_1(l_1, l_2) &= \frac{1}{96} l_2(16\pi^2 + 2l_1^2 + 2l_2^2) \\
\left.\frac{d}{dl_2}\right|_{2\pi i} V_1(l_1, l_2) &= \frac{2\pi i}{96}(8\pi^2 + 2l_1^2) \\
&= \frac{2\pi i}{4.24}(4\pi^2 + l_1^2) = \frac{2\pi i}{4} V_1(l_1)
\end{aligned}
$$

Motivation: Do,Norbury

Weil-Petersson volumes and cone surfaces, (2005)

- Interpolating/smoothing the forgetful map $(\Sigma_g, p) \to \Sigma_g$
- Studying degeneration of associated fibration(s)

$$\Sigma_g^{\sim} \to \mathcal{T}(\Sigma_{g,1}) \to \mathcal{T}(\Sigma_g)$$

$$\Sigma_g \to \mathcal{T}(\Sigma_{g,1})/\mathcal{MCG} \to \mathcal{T}(\Sigma_g)/\mathcal{MCG}$$

What happens to

- the topology
- the dynamics of $\mathcal{MCG}$

as we move through the corresponding relative character varieties?

Representations are not discrete, faithful

Lemma ((Generic) Hyperbolicity)

If $\rho : \pi_1 \to SL_2(\mathbb{R})$ discrete, faithful then $\rho(\gamma)$ is hyperbolic $\forall\gamma(\neq 1) \in \pi_1$.

Definition: Simple Hyperbolicity

$\forall\gamma \in \pi_1$ essential, simple loop $\rho(\gamma)$ hyperbolic.

Definition: Simple Spectrum

Simple spectrum $:= \{|\mathrm{tr}\,\rho(\gamma)|, \gamma$ essential, simple loop$\}$.

Ultimate aim

want to show that the simple spectrum grows as quickly as for a discrete representation.

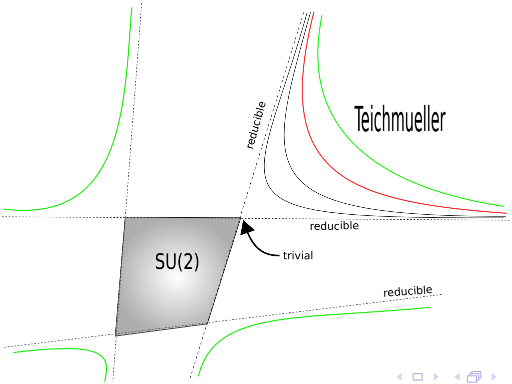
Σ = torus with a hole/cone point

Relative character variety

$\pi_1 = \langle \alpha, \beta \rangle$, $\delta = [\alpha, \beta]$ peripheral

$$\mathcal{T}(\Sigma)_\theta \leftrightarrow \text{component of the rel. character variety} \hookrightarrow \mathbb{R}^3$$

$$\{\rho,\ |\mathrm{tr}\,\rho(\delta)| = 2\cos(\tfrac{\theta}{2})\}/SL_2(\mathbb{R}) \qquad \begin{pmatrix} \mathrm{tr}\,\rho(\alpha) \\ \mathrm{tr}\,\rho(\beta) \\ \mathrm{tr}\,\rho(\beta\alpha) \end{pmatrix}$$



Σ = torus with a hole/cone point

$\mathcal{T}(\Sigma)_\theta \leftrightarrow$ component of the relative character variety.

- Tan, Wong, Zhang: *Generalizations of McShanes identity to hyperbolic cone-surfaces* (2008).
- Tan, Wong, Zhang: *The SL(2,C) character variety of the one-holed torus*(2005)
- Goldman, *The modular group action on the real SL(2)-characters of a one-holed torus*

Theorem (Goldman, Tan-Wong-Zhang)

For $\theta < 2\pi$:

- *The action of $\mathcal{MCG}$ is proper on $\mathcal{T}(\Sigma)_\theta$.*
- *If $\rho \in \mathcal{T}(\Sigma)_\theta$ and γ an essential simple loop then $\rho(\gamma)$ is hyperbolic $\Rightarrow \exists \ell_\gamma > 0$, $|\mathrm{tr}\,\rho(\gamma)| = 2\cosh(\ell_\gamma/2)$*
- *The "Markoff map" satisfies "Fibonacci growth"*

Σ = torus with a hole/cone point

(Bowditch (1993),Sakuma, Tan-Wang-Zhang)

Markoff map

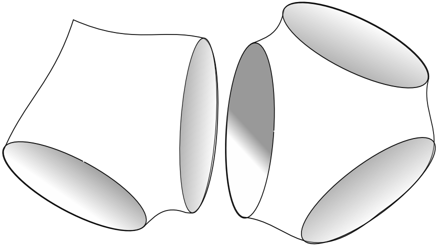
A particular enumeration (associated to a binary tree)
of the (traces of) simple geodesics in a punctured torus group.

Fibonacci growth

$\exists \epsilon > 0$ such that $\forall \gamma$ essential simple loop $\ell_\gamma \geq \epsilon l_\gamma$.
l_γ = length of geodesic homotopic to γ
for a fixed complete hyperbolic metric on Σ.

Geometry breaks down: Pants decomposition

Σ = Four-holed sphere = (pair of pants) $\sqcup$ (pair of pants) / $\sim$



Theorem (Fenchel-Nielsen coordinates)

$$\begin{array}{ccc} \mathbb{R}^+ \times \mathbb{R} & \to & \mathcal{T}(\Sigma) \\ (l_\alpha, \tau_\alpha) & \mapsto & \rho_{(l,\tau)} \end{array}$$

is a diffeomorphism.

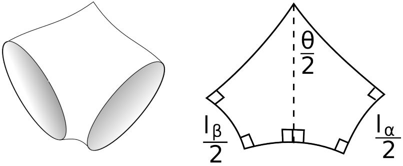
(De)Motivation

Dryden-Parlier: Collars and partitions of hyperbolic cone-surfaces. (2007)

Abstract For compact Riemann surfaces, the collar theorem and Bers partition theorem are major tools for working with simple closed geodesics. The main goal of this article is to prove similar theorems for hyperbolic cone-surfaces. Hyperbolic two-dimensional orbifolds are a particular case of such surfaces. We consider all cone angles to be strictly less than π to be able to consider partitions.

Pair of pants with a large cone angle

Pair of pants = double of (generalised) right hexagon
= double of (pair of isometric (generalised) right pentagons).



Lemma

The possible values of $\ell_\alpha \leq \ell_\beta$ satisfy

$$1 \geq \frac{\cosh(\ell_\alpha/2)}{\cosh(\ell_\beta/2)} \geq -\cos(\theta/2) > 0,\ \theta > \pi.$$

Seems impossible:

either to prove Collar Lemma
or find pants decomposition.

(Cone) Point of view

Definition: Simple Hyperbolicity

$\forall \gamma \in \pi_1$ essential, simple loop $\rho(\gamma)$ hyperbolic.

Fibonacci growth

$\exists \epsilon > 0$ such that $\forall \gamma$ essential simple loop $\ell_\gamma \geq \epsilon l_\gamma$.
$l_\gamma =$ length of geodesic homotopic to γ
for a fixed complete hyperbolic metric on Σ.

Strategy

Simple hyperbolicity $\Rightarrow \exists$ Fenchel-Nielsen coordinates.

Boundary: Pants decomposition + Collar Lemma
$\Rightarrow \exists$ model paths for simple loops $\subset \Sigma$

Cone points: Fenchel-Nielsen coordinates + "Collar Lemma"
$\Rightarrow \exists$ model paths for simple loops $\subset \Sigma^\sim = \mathbb{H}$

Compare model paths $\Rightarrow$ Fibonacci growth.
$\Rightarrow \mathcal{MCG}$ acts prop. disc

Four-holed sphere

Prior art

- Goldman-Benedetto: The topology of the relative character varieties of a quadruply-punctured sphere
- Cantat-Loray: Holomorphic dynamics, Painlevé VI equation, and character varieties
- Gauglhofer-Semmler, Trace coordinates of Teichmuller space of Riemann surfaces of signature

Do-Norbury

- fix three boundary lengths l_1, l_2, l_3
- vary the fourth $l_4 = i\theta$, $\theta \in [0, 2\pi[$
- $\theta = 0$ get a surface with a cusp
  $\rho : \pi_1 \to SL_2(\mathbb{R})$ is discrete, faithful.

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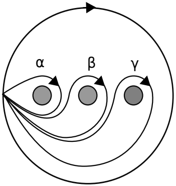
Starting from scratch : 3-punctured sphere + cone point

Four holed sphere: $\pi_1 = \langle \alpha, \beta, \gamma, \delta,\ \delta = \gamma\beta\alpha \rangle$.



- fix boundary lengths $l_1 = l_2 = l_3 = 0 \Rightarrow \alpha, \beta, \gamma$ parabolic.
- vary the fourth $l_4 = i\theta \Rightarrow |\mathrm{tr}\,\delta| = 2\cos(\theta/2))$

$$\begin{array}{ccccccc}
(X \subset \mathbb{R}^3) & \hookrightarrow & \mathrm{Hom}(\pi_1, SL_2(\mathbb{R})) & \to & \mathcal{T}(\Sigma)_\theta \\
(a, b, c) & \mapsto & (\alpha, \beta, \gamma) \in (SL_2(\mathbb{R}))^3 & \mapsto & \rho_{abc}
\end{array}$$

$$\alpha(0) = 0$$
$$\beta(1) = 1$$
$$\gamma(\infty) = \infty$$

Explicit matrices for ρ_{abc}

$$\gamma = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1+b & -b \\ b & 1-b \end{pmatrix}, \alpha = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

$X := \{(a, b, c), \operatorname{tr}\delta = 2\cos(\theta/2)\}$ = graph over $\Omega \subset \mathbb{R}^2$.

$$\begin{array}{rcl} \Omega := \{ab - a - b > 0, a > 1, b > 1\} & \to & X \\ (a, b) & \mapsto & (a, b, c(a, b)) \end{array}$$

If you prefer $X \to \Omega, (a, b, c) \mapsto (a, b)$ is a chart.

Calculation I

$$\operatorname{tr}\delta = \operatorname{tr}\gamma\beta\alpha = 2 - ab - ac - bc + abc = 2\cos(\theta/2)$$

$$\Rightarrow c = c(a, b) = \frac{2\cos(\theta/2) - (2 - ab)}{ab - a - b}$$

Surjectivity of $\Omega \to \mathcal{T}(\Sigma)_\theta$

Aim show the $(a,b) \mapsto \rho_{abc} \in \mathcal{T}(\Sigma)_\theta$ is a diffeo

$$\begin{array}{ccccccc} \Omega & \to & X & \to & \mathcal{T}(\Sigma)_\theta & \to & \mathbb{R}^3 \\ (a,b) & \mapsto & (a,b,c(a,b)) & \mapsto & \rho_{abc} & \to & (\operatorname{tr}\beta\gamma, \operatorname{tr}\beta\alpha, \operatorname{tr}\alpha\gamma) \end{array}$$

Calculation II

$$\operatorname{tr}\beta\alpha = 2 - ab,\ \operatorname{tr}\gamma\alpha = 2 - ac,\ \operatorname{tr}\beta\gamma = 2 - bc$$

these formulae $\Rightarrow$ ρ_{abc} injective + (local) diffeo.

Lemma (Simple hyperbolicity on image)

If $\theta \neq 2\pi$ then $\operatorname{tr}\beta\alpha = 2 - ab < -2$ on Ω.

Surjectivity of $\Omega \to \mathcal{T}(\Sigma)_\theta$

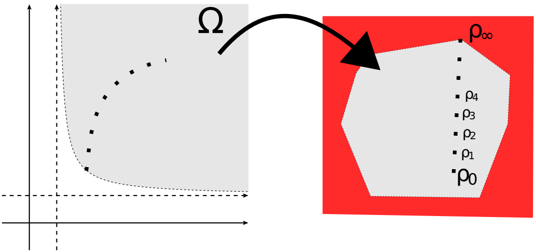
Since $c(a,b) = \frac{2\cos(\theta/2)-(2-ab)}{ab-a-b} = \frac{>(2\cos(\theta/2)+2)\geq 0}{(>0)}$

Lemma (Properness)

If (a_n, b_n) leaves compact sets in Ω then

- *at least one of $a_n, b_n, c(a_n, b_n) \to \infty$.*
- *at least one of* $\operatorname{tr}\beta\alpha = 2 - ab,\ \operatorname{tr}\gamma\alpha = 2 - ac,\ \operatorname{tr}\beta\gamma = 2 - bc \to -\infty.$

$\Rightarrow$ Surjectivity

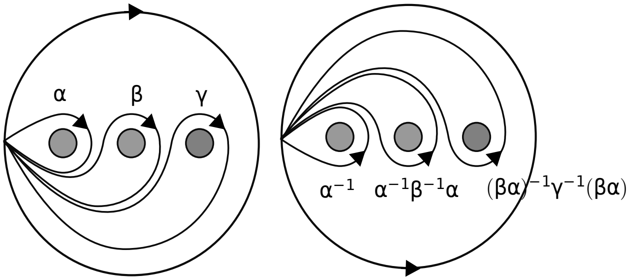


$\mathcal{MCG}$ dynamics on Ω

$\exists$ three involutions $\Omega \to \Omega$

- induced by homeomorphisms of the four holed sphere
- that preserve $2 - ab - ac - bc + abc$

$$I_\alpha := \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a/(a-1) \\ b(a-1) \\ c(a-1). \end{pmatrix}, \quad \begin{array}{ccc} \alpha & \mapsto & \alpha^{-1} \\ \beta & \mapsto & \alpha^{-1}\beta^{-1}\alpha \\ \gamma & \mapsto & (\beta\alpha)^{-1}\gamma^{-1}(\beta\alpha) \end{array}.$$



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Observation

Involutions are good because the dynamics is determined by configuration of fixed point sets.

$$(2, b, c) \mapsto (2/(2-1), (2-1)b, (2-1)c) = (2, b, c)$$

so fixed point set of I_α is $\{a = 2\}$

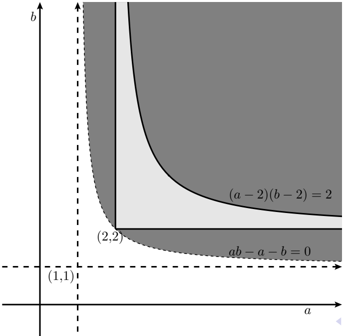
$\mathcal{MCG}$ dynamics: fixed point sets of involutions

Observation

Involutions are good because fixed point sets determine a fundamental domain.

Fixed point sets of involutions $I_\alpha, I_\beta, I_\gamma$

$\{a = 2\}$, $\{b = 2\}$
$\{c(a,b) = 2\} = \{(a-2)(b-2) = 2\cos(\frac{\theta}{2}) + 2\}$.



$\mathcal{MCG}$ dynamics: volume

Theorem (Nakanishi-Naatanen, M.)

- *$I_\alpha, I_\beta, I_\gamma$ generate a group isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$*
- *this group contains $\mathcal{MCG}$ as a finite index subgroup.*
- *$\{(a, b) : a > 2,\ b > 2,\ c(a, b) > 2\}$ is a fundamental domain.*

(Wolpert's) symplectic form on $\mathcal{T}(\Sigma)_\theta$

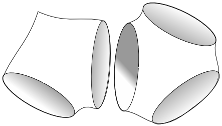
$$\omega_{WP} = \frac{da \wedge db}{ab - b - a} = \frac{db \wedge dc}{bc - b - c} = \frac{dc \wedge da}{ac - a - c}$$

$I_\alpha^*\ da \wedge db = d\left(\frac{a}{a-1}\right) \wedge d((a-1)b) = \frac{-da \wedge db}{(a-1)}$

$I_\alpha^*(ab - a - b) = \frac{ab-a-b}{a-1}$

Geometry breaks down: Pants decomposition

Σ = Four-holed sphere = (pair of pants) $\sqcup$ (pair of pants) / $\sim$



Theorem (Fenchel-Nielsen coordinates)

For the three-holed sphere + cone point

$$\begin{array}{ccc} \mathbb{R}^+ \times \mathbb{R} & \to & \mathcal{T}(\Sigma)_\theta \\ (l_{\beta\alpha}, \tau_{\beta\alpha}) & \mapsto & \rho_{(l,\tau)} \end{array}$$

is a diffeomorphism.

Write $\beta\alpha^\pm$ = attracting/repelling fixed points, $\alpha^+ = 0, \gamma^+ = \infty$

$$\operatorname{tr} \beta\alpha = 2 - ab, \tau = \log \left| \frac{\alpha^+ - \beta\alpha^+}{\alpha^- - \beta\alpha^-} . \frac{\gamma^+ - \beta\alpha^-}{\gamma^- - \beta\alpha^+} \right| = \log \left| \frac{0 - \beta\alpha^+}{0 - \beta\alpha^-} \right|$$

The end?

$$2\cosh(l_{\beta\alpha}) = \mathrm{tr}\,\beta\alpha = 2 - ab$$
$$\tau = \log\left|\frac{\beta\alpha^+}{\beta\alpha^-}\right|$$

On Ω

- $l_{\beta\alpha} \in \mathbb{R}^+$ well defined ($\mathrm{tr}\,\beta\alpha =$ const. hyperbolae foliate Ω).
- $\beta\alpha^+ \neq \beta\alpha^- \notin \{0, 1, \infty\}$

Theorem (M.)

For the three-holed sphere + cone point the volume of the moduli space is given by the volume polynomial provided $\theta < 2\pi$.

Proof: Volume $= \int_2^\infty \int_2^c \frac{dadb}{ab-a-b}$, $c = \frac{z+2}{ab-a-b}$ analytic for $z \in \mathbb{C}$ gives volume for $z = 2\cosh(l/2)$ so for $z = 2\cos(\theta/2)$ too.