AdS geometry as a tool for Teichmüller theory

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Goals of this talk

- Advertise 3d AdS geometry as a tool for Teichmüller theory,
- Explain basics of AdS geometry,
- State some recent results obtained using AdS,
- Examples of proofs,
- Some open questions here and there.

AdS_3

Recall that

$$H^3 = \{x \in \mathbb{R}^{3,1} \mid \langle x, x \rangle = -1 \& x_0 > 0\} .$$

$$AdS_3 = \{x \in \mathbb{R}^{2,2} \mid \langle x, x \rangle = -1\} .$$

Lorentz analog of H^3 : complete, constant curvature -1.
From relativity : "Anti de Sitter", model for gravity (no matter).
Lorentz analog of S^3 : $PSL(2, \mathbb{R})$ w/ Killing metric, isometry group, etc
Basic idea : hyperbolic and AdS 3-mflds as tools for Teichmüller theory.

Hyperbolic 3-manifolds and Teichmüller theory

Based (mostly) on quasifuchsian 3-manifolds. Examples of applications include :

- complex projective structures on surfaces,
- complex earthquakes (McMullen),
- the volume of the convex core of quasifuchsian manifolds is coarsely equivalent to the Weil-Petersson distance between the metrics on its boundary (Brock),
- the renormalized volume as a Kähler potential for WP,
- properties of the grafting map.

Not developed here.

AdS 3-manifolds and Teichmüller theory

Some aspects :

- earthquakes,
- extensions of the earthquake flow,
- minimal Lagrangian diffeos.

AdS side involves physically relevant notions :

- globally hyperbolic (GH) spaces (analogs of quasifuchsian),
- "particles",
- multi-black holes,
- maximal surfaces.

Notations : S closed surface of genus ≥ 2, $\mathcal{T}$ Teichmüller space.

Measured laminations

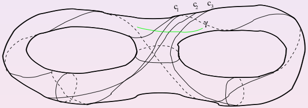
$\mathcal{WM}$ = { weighted multicurves on S } : set of disjoint simple closed curves, each with a positive weight.

$\mathcal{WM}$ is infinite : simple closed curves on S can wrap around a lot.

Let $(c_i, l_i)_{i=1,\cdots,n} \in \mathcal{WM}$, the c_i form a *lamination* and the l_i define a *transverse measure* : gives a total weight to γ, transverse to the c_i.

This gives a topology to $\mathcal{WM}$.

The completion of $\mathcal{WM}$ is the space of *measured laminations* $\mathcal{ML}$.

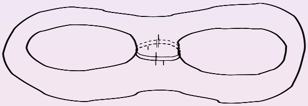


Measured laminations can be pretty complicated.

- $\mathcal{ML} \simeq \mathbb{R}^{6g-6}$.
- $\partial\mathcal{T} \simeq \mathcal{ML}/\mathbb{R}_{>0}$ (Thurston).
- $\mathcal{T} \times \mathcal{ML} \simeq T^*\mathcal{T}$.

Thurston's Earthquake Thm

Start with a hyperbolic surface.
If $w \in \mathcal{ML}$ is a weighted curve and $h \in \mathcal{T}$, $E_l(w)(h)$ is obtained by realizing w as a geodesic in h, cutting S open along w, turning the left-hand side by the weight, and gluing back.
Defines a homeomorphism



$$E_l(w) : \mathcal{T} \to \mathcal{T} \ .$$

Extends by continuity to $E_l : \mathcal{T} \times \mathcal{ML} \to \mathcal{T}$ (Thurston).
Thm (Thurston, Kerckhoff). $\forall h, h' \in \mathcal{T}, \exists! \lambda \in \mathcal{ML}, h' = E_l(\lambda)(h)$.
Simple proof by Mess (1990) based on AdS geometry.

Extensions of the Earthquake Thm

Extension of the Earthquake Theorem :

- to hyperbolic surfaces with cone sings of angle $< \pi$. (w/ Francesco Bonsante.)
- to hyperbolic surfaces with geodesic boundary : 2^N earthquakes sending h to h'. (w/ Bonsante, Kirill Krasnov).

The proof of the 1st statement is equivalent to an extension of the Mess parameterization for GH AdS manifolds with "particles" : cone singularities along time-like lines, $\theta < \pi$. The analoguous quasifuchsian statement holds : Bers-type theorem for quasifuchsian manifolds with cone singularities of $\theta < \pi$ along infinite lines (Lecuire, Moroianu, S.).
The 2nd statement is based on *multi-black holes* : like globally hyperbolic manifolds, based on a complete, non-compact surface. AdS analogs of Schottky mflds.

Dynamics of earthquakes

Thm (Bonsante, S.). Let $\lambda, \mu \in \mathcal{ML}$ that fill S. Then $E_r(\lambda) \circ E_r(\mu)$ has a fixed point on $\mathcal{T}$.
Uniqueness ?
See talk by Francesco.

A cyclic extension of the earthquake flow

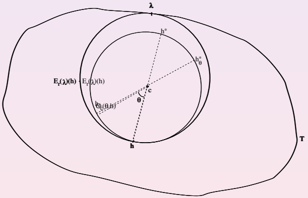
For $\lambda \in \mathcal{ML}$ fixed, $E_l(\lambda)$ defines an action of $\mathbb{R}$ on $\mathcal{T}$, by $(t, h) \mapsto E_l(t\lambda)(h)$. Analog of horocyclic flow.

We define (w/ Bonsante & Gabriele Mondello) an "extension" : equivalently

- for $c \in \mathcal{T}$, $C_c : S^1 \times \mathcal{T} \to \mathcal{T}$,
- an action D of S^1 on $\mathcal{T} \times \mathcal{T}$.

3 (related) definitions based on

- GH AdS 3-mflds,
- minimal Lagrangian maps,
- holomorphic quadratic differentials.



Properties of the cyclic flow

Some properties :

- Limits to the earthquake flow : if $t_n h_n^* \to \lambda$ then $D_{t_n}(h, h_n^*) \to E_l(\lambda/2)(h)$.
- Extension of the earthquake thm :
$$\forall \theta \in S^1 \setminus \{0\}, \forall h, h' \in \mathcal{T}, \exists ! c \in \mathcal{T}, C_c(\theta, h) = h' \ .$$
- Has a complex extension, which limits to McMullen's complex earthquakes.
- Extends to a S^1 action on the universal Teichmüller space.

The extension of the Earthquake Thm follows from a recent result of Barbot, Béguin and Zeghib on constant Gauss curvature foliations of AdS manifolds.

The universal Teichmüller space

A homeo of S^1 is *quasi-symmetric* if it is the boundary of a quasi-conformal diffeo of the disk.
Def. $\mathcal{T}_U =$ space of quasi-symmetric orientation-preserving homeos of S^1, up to $PSL(2,\mathbb{R})$.
Let $\rho_0 \in \mathcal{T}$, then any $\rho \in \mathcal{T}$ is conjugated to ρ_0 by a quasi-conformal diffeo ϕ. Moreover $\partial\phi$ is unique. Therefore all $\mathcal{T}$ embed in $\mathcal{T}_U$.
Question : canonical quasi-conformal extension(s) to the disk of a quasi-symmetric homeo ?
Conj (Schoen). Any quasi-symmetric homeo of S^1 has a unique quasi-conformal harmonic extension to the disk.
Uniqueness. Partial results on existence. True for closed surfaces.

Extensions of quasi-symmetric homeos

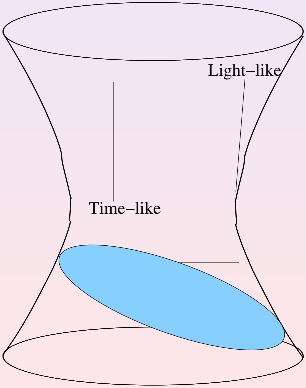
Def. a diffeo $\phi : H^2 \to H^2$ is *minimal Lagrangian* iff it is area-preserving and its graph is minimal in $H^2 \times H^2$.
ϕ is min Lagrangian iff $\phi = v \circ u^{-1}$, where $u, v : D \to H^2$ are harmonic maps with opposite Hopf differentials. "Squares" of harmonic map.
Thm (Bonsante, S). any quasi-symmetric homeo h of S^1 has a unique extension as a quasi-conformal minimal Lagrangian diffeo of H^2.
Known (Schoen, Labourie 1992) for closed surfaces. Also when h has small dilation (Aiyama, Akutagawa, Wan 2000).

AdS_3 as a Lorentz analog of H^3

$$AdS_3 = \{x \in \mathbb{R}^{2,2} \mid \langle x, x \rangle = -1\} .$$

Constant curvature -1, $\pi_1(AdS_3) = \mathbb{Z}$.

- Conformal model, in a cylinder.
- Projective model, in a quadric.
- Space-like, time-like, light-like directions. Time-like geodesics are closed of length 2π.
- Totally geodesic space-like planes $\simeq H^2$.
- $Isom(AdS_3) = O(2,2)$.
- Boundary at ∞ with Lorentz-conformal structure.

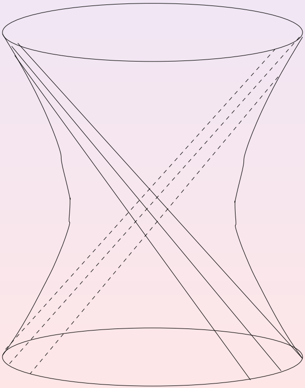


AdS_3 as a Lorentz analog of S^3

Recall : $S^3 = SU(2) \simeq SO(3)$, and $Isom(S^3) = O(4) \simeq O(3) \times O(3)$.
$AdS_3 = PSL(2, \mathbb{R})$ with its Killing metric. Left and right actions of $PSL(2, \mathbb{R})$, identifies $Isom_0(AdS_3) = PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ (up to index 2).

Geometrically :

- $\partial_\infty AdS_3$ is foliated by 2 families of lines.
- Thus $\partial_\infty AdS_3 \simeq \mathbb{R}P^1 \times \mathbb{R}P^1$,
- Isometries act projectively on each family,
- Space-like curves in $\partial_\infty AdS_3$ are graphs of functions $\mathbb{R}P^1 \to \mathbb{R}P^1$.



Globally hyperbolic AdS manifolds

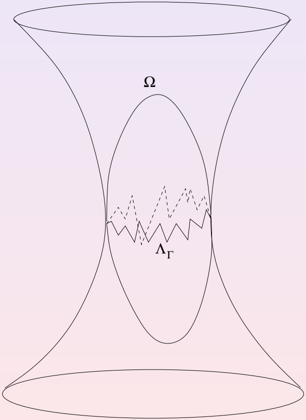
Def. an AdS mfld M is *maximal globally hyperbolic* if

- it contains a closed, space-like surface S,
- any inextendible time-like curve intersects S exactly once,
- it is maximal (for inclusion) under those properties.

Then $M \simeq S \times \mathbb{R}$, and $M = \Omega/\rho(\pi_1 S)$, where $\Omega \subset AdS_3$.
GH AdS mflds are strongly reminiscent of quasifuchsian hyperbolic mflds, but in a way more relevant to Teichmüller theory (Mess).

GHMC AdS vs quasifuchsian

M has a "limit set" Λ_Γ, which is a Jordan curve. $\Lambda_\Gamma = \partial\Omega \cap \partial_\infty AdS_3$.
M has a "convex core", $C(M) = CH(\Lambda_\Gamma)/\Gamma$.
It has two boundary components, both with hyperbolic induced metrics $m_\pm$, bent along measured laminations $l_\pm$ that fill (Mess).
Question (Mess). can any $m_\pm$ be uniquely realized ?
Existence seems to hold (Boubacar Diallo, in progress). *Uniqueness ?*
Thm (Bonsante, S.) Any l_-, l_+ that fill can be realized. *Uniqueness ?*



A Bers-type parametrization

Given a GHMC AdS mfld M, $\rho : \Gamma \to SO(2,2) \simeq PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$.
So, $(\rho_L, \rho_R) : \Gamma \to PSL(2,\mathbb{R})$.
Thm (Mess).

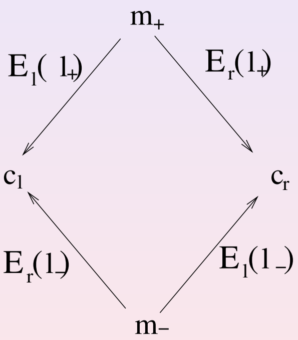
- ρ_L, ρ_R have maximal Euler number.
- The map $GH \to \mathcal{T} \times \mathcal{T}$ is a homeomorphism.

The hyperbolic metrics c_L, c_R corresponding to ρ_L, ρ_R are analogs of the conformal metrics at infinity.

Proof of the Earthquake Thm

$m_\pm$ are related to c_l, c_r by earthquakes along $l_\pm$. The Earthquake thm follows from this by simple arguments.

- Fix c_l, c_r. By Mess' thm, there are unique $m_\pm, l_\pm$.
- $c_r = E_r(l_+) \circ E_l(l_+)^{-1}(c_l)$
- $E_l(l_+)^{-1} = E_r(l_+)$,
- so $E_r(l_+) \circ E_l(l_+)^{-1} = E_r(2l_+)$.
- Thus $c_r = E_r(2l_+)(c_l)$, and similarly $c_r = E_l(2l_-)(c_l)$.
- Uniqueness follows from the same argument.



The existence of fixed points of $E_l(\lambda) \circ E_l(\mu)$ follow similarly from prescribing l_-, l_+.

Maximal surfaces in AdS

Let $\Sigma \subset AdS_3$ be a space-like graph. We call :

- I the induced metric, J its complex structure,
- B the shape operator, $BX = -\nabla_X N$,
- E the identity.

Def. $h_L, h_R = I((E \pm JB)\cdot, (E \pm JB)\cdot)$.
Prop (Krasnov, S.). if Σ has principal curvatures $|k_i| < 1$ then h_L, h_R are hyperbolic metrics. If h_L, h_R are complete, we obtain $\phi : H^2 \to H^2$.
Related to the left/right representations for GH mflds.
Prop. Σ is *maximal* iff ϕ is min Lagrangian. It is quasi-conformal iff $|k_i| < 1$ uniformly.
Prop. If in addition $\partial_\infty \Sigma$ is the graph of a quasi-symmetric homeo $\subset \partial_\infty AdS_3 \simeq \mathbb{R}P^1 \times \mathbb{R}P^1$, then h_L, h_R are complete and $\partial_\infty \Sigma$ is the graph of ϕ.

Statement on maximal surfaces

Thm B (Bonsante,S). let $\Gamma \subset \partial_\infty AdS_3$ be the graph of a quasi-symmetric homeo. Then there exists a unique maximal surface $\Sigma \subset AdS_3$ with $|k_i| < 1$ uniformly such that $\partial_\infty \Sigma = \Gamma$.
Thm A follows through the correspondance with min Lagrangian maps.
Thm B has a partial extension to higher dimensions (existence).
The key step in the proof of Thm B are compactness estimates for maximal surfaces in AdS_n, using results of Barnik (1984).

Questions

AdS geometry and its applications to Teichmüller theory remains relatively open.

- Open questions on the boundary of the convex core of GH mflds, and applications to earthquakes.
- Use AdS to prove Schoen's conjecture on harmonic extensions ?
- Extend to AdS setting various results known for quasifuchsian mflds ?
- Other questions and applications, not yet discovered ? ?

The end

Thanks for your attention !