Regenerating hyperbolic cone 3-manifolds from dimension 2

J. Porti

Universitat Autònoma de Barcelona

Geometry, Topology and Dynamics of Character Varieties

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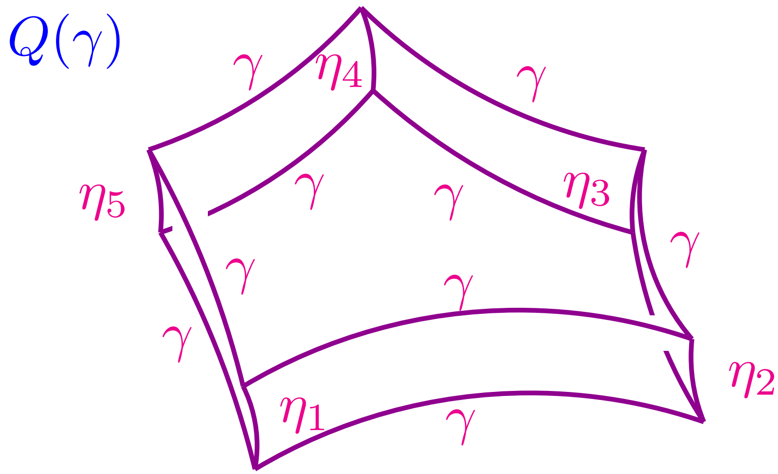

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$$2\gamma + \eta_j > \pi$$

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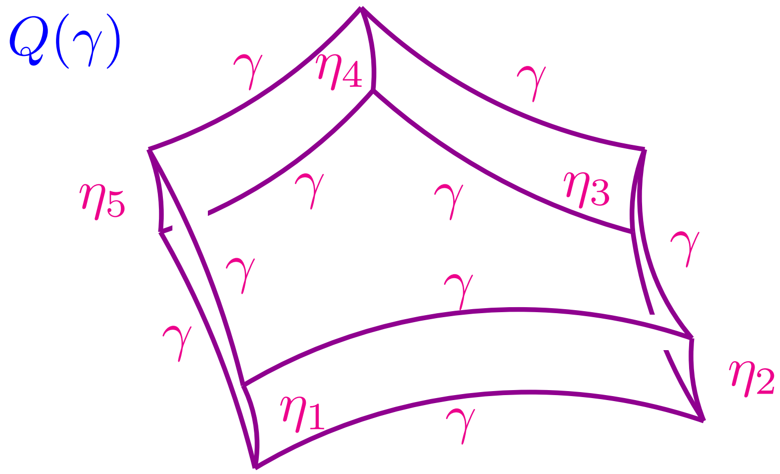
Question: Fixing the η_i, what is GH-$\lim_{\gamma \to \frac{\pi}{2}} Q(\gamma)$?

Degenerating a hyperbolic prism



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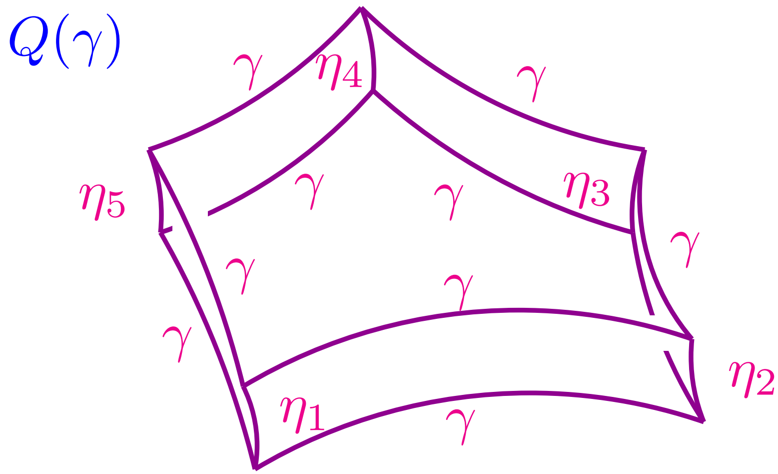

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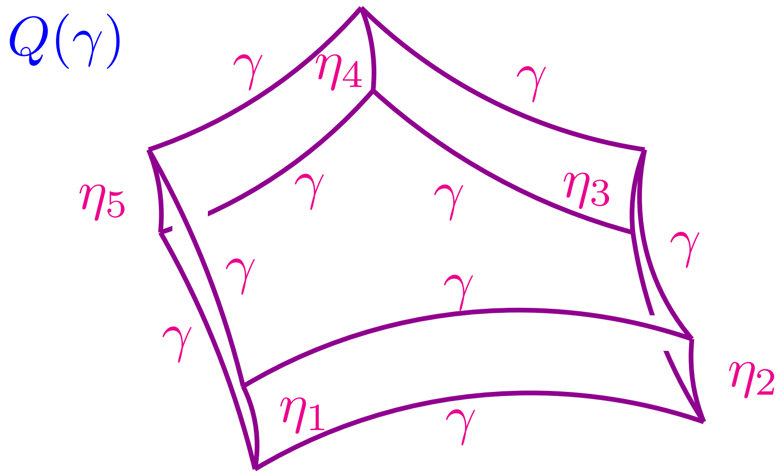

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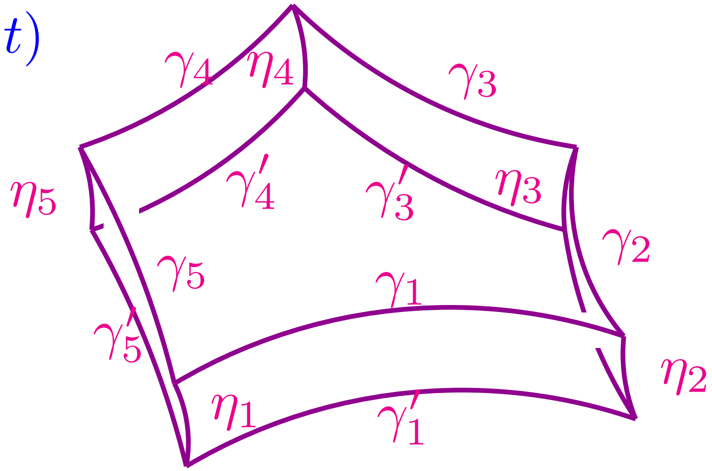
- $Q(\gamma) \cup_\partial \overline{Q}(\gamma)$ 3-dim hyperbolic cone mfld. with cone angles $2\eta_i$, 2γ.
- Existence of polygon with minimal perimeter: use earthquake thm for cone surfaces with cone angles $< \pi$ (Bonsante-Schlenker).

Change the speed of the γ_i's

$\gamma_i = \gamma_i' = \frac{\pi}{2} - w_i t$, $t \searrow 0$, for some $w_i > 0$

$Q(t)$



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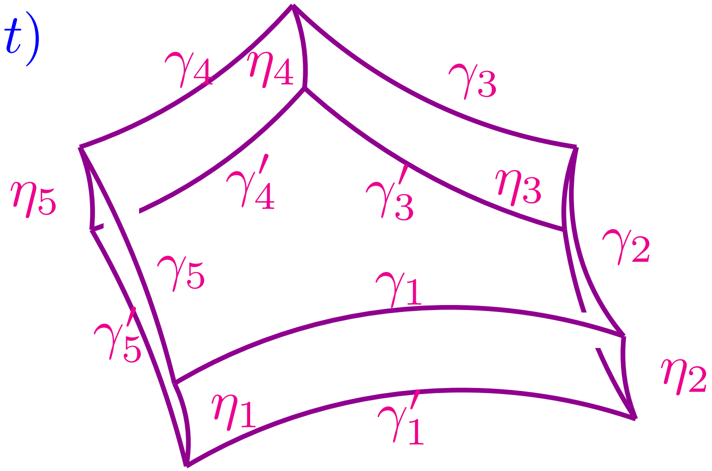

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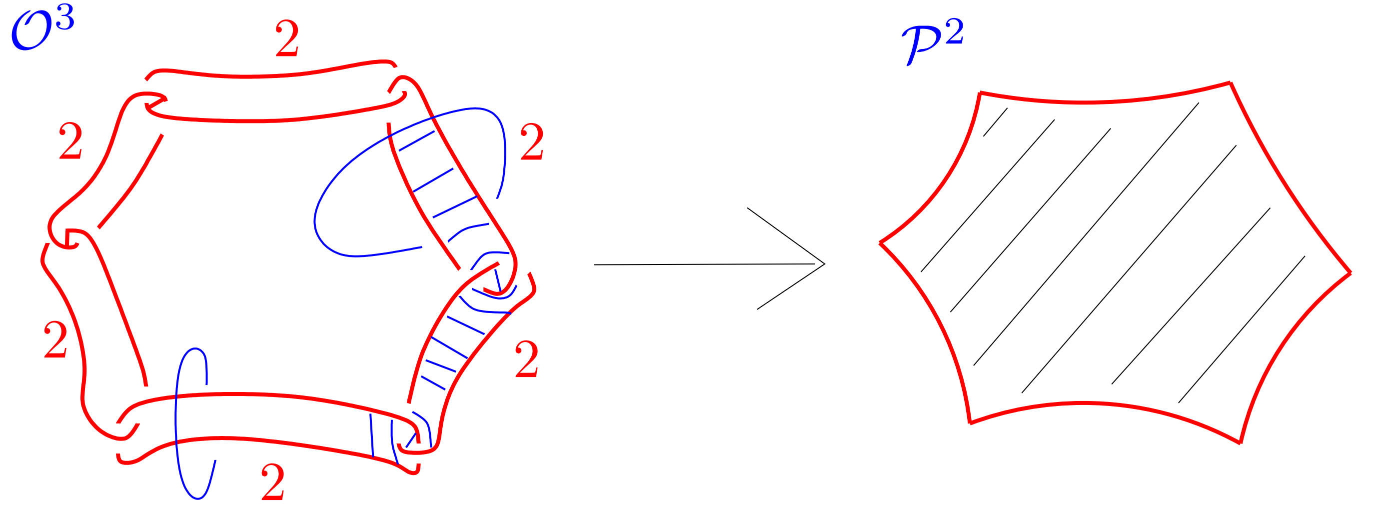
GH-$\lim_{t \to 0} Q(t) =$ polygon with angles $\eta_1, \ldots, \eta_n$ that minimizes

$$\sum_i w_i \,\mathrm{length}(\text{edge}_i)$$

- Orbifold Seifert fibration $\mathcal{O}^3 \to \mathcal{P}^2$



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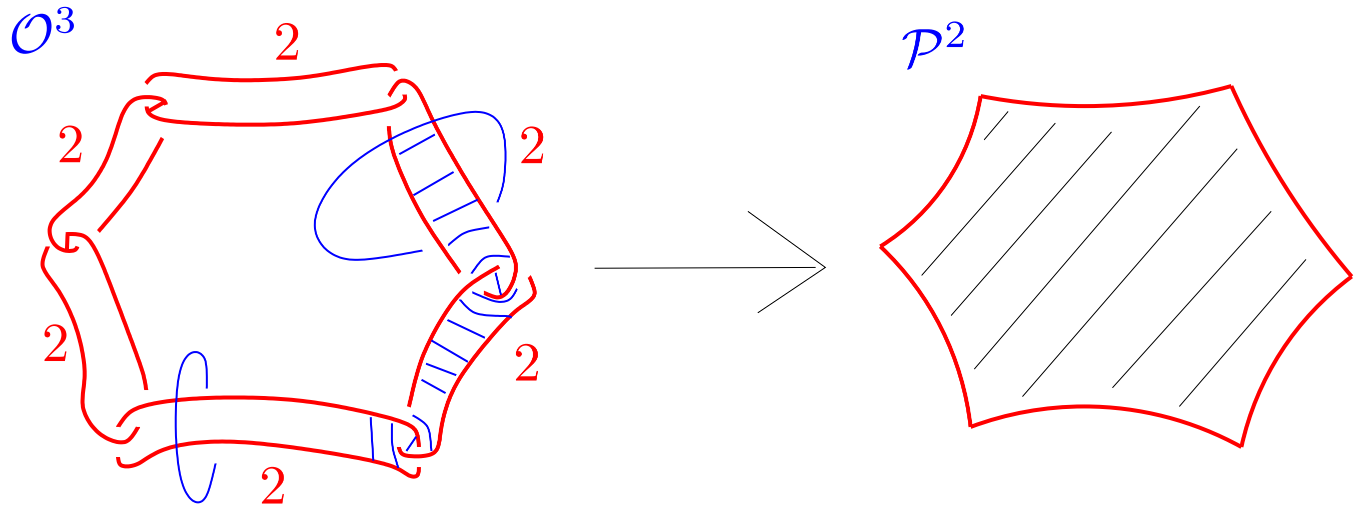

$\mathcal{P}^2$ is a polygonal Coxeter orbifold with mirrored edges and corners.

- $Sing(\mathcal{O}^3) \to Sing(\mathcal{P}^2)$ is 2:1
- Fibers: S^1 or $S^1/\mathbf{Z}_2 =$ interval with boundary
- fiber of "Seifert order" $q_i \geq 1$ with ramification index $k_i \geq 1$ $\Big\} \leftrightarrow$ corners of $\mathcal{P}$ of angle $\dfrac{\pi}{k_i q_i}$.

- Orbifold Seifert fibration $\mathcal{O}^3 \to \mathcal{P}^2$
- $\mathcal{P}^2$ hyperbolic Coxeter orbifold with angles $\frac{\pi}{k_1 q_1}, \ldots, \frac{\pi}{k_n q_n}$.

Theorem

For $\alpha \in (\pi - \varepsilon, \pi)$ there is $C(\alpha)$ a hyperbolic cone structure on $|\mathcal{O}^3|$, $|C(\alpha)| = |\mathcal{O}^3|$, $Sing(C(\alpha)) = Branch(\mathcal{O}^3)$, vertical cone angles $\frac{2\pi}{k_i}$ and horizontal cone angle α, so that

GH-$\lim_{\alpha \to \pi^-} C(\alpha) =$ polygon of minimal perimeter with angles $\frac{\pi}{k_1 q_1}, \ldots, \frac{\pi}{k_n q_n}$

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- View $\mathcal{P}^2$ as a degenerated structure
- The interior of $\mathcal{P}^2$ can have one cone point ($\exists \widetilde{\mathcal{P}^2}$ smooth polygon) but not two ($\mathcal{O}^3 \setminus Sing(\mathcal{O}^3)$ would contain essential tori)

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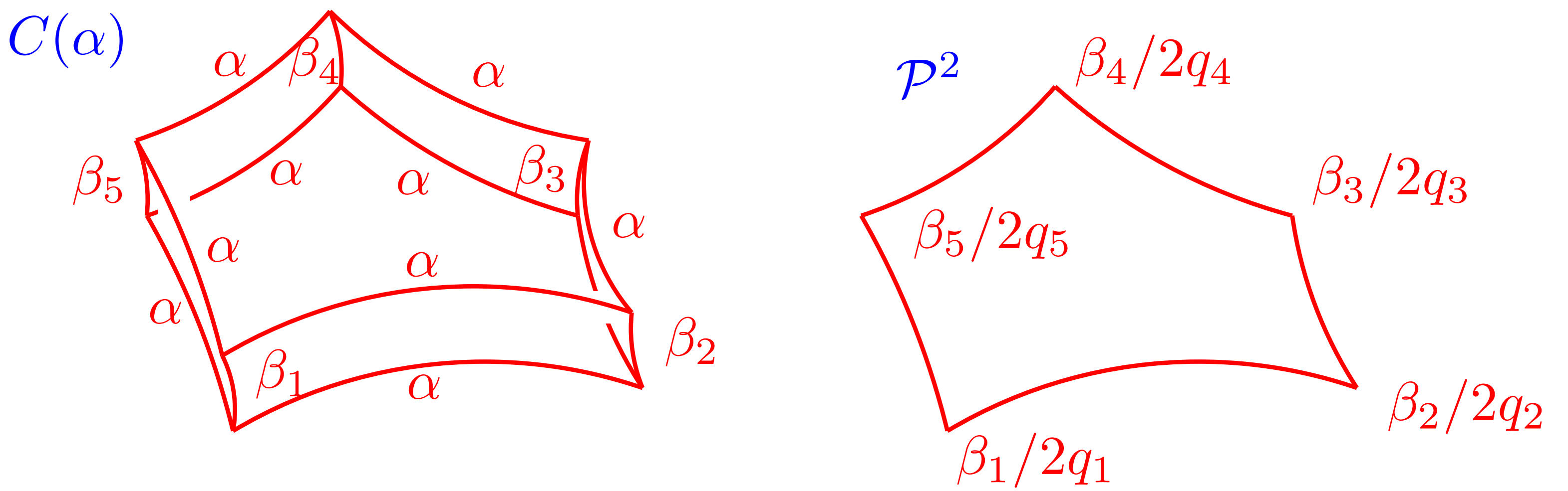
- Can assume also that the singular fibres have a (fixed) cone angle, combinig cone manifolds and orbifold fibrations.

In this talk I shall assume all $k_i > 1$.

Namely: all singular fibers are in the branch locus of $\mathcal{O}^3$



$0 < \beta_j < 2\pi$ fixed $\quad \beta_j/2q_j \leq \pi/2$, $\sum(\pi - \beta_j/2q_j) > 2\pi$

Goal: to put a hyperbolic cone structure for $\pi - \varepsilon < \alpha < \pi$ so that

$C(\alpha) \underset{GH}{\longrightarrow} \mathcal{P}^2$ of minimal perimeter.

Developing map

$$\mathcal{O}^3 \to \mathcal{P}^2$$

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Non-complete structures on $M = \mathcal{O}^3 \setminus Sing$=handlebody

$$Dev_0 : \widetilde{M} \to \widetilde{\mathcal{P}^2 \setminus vertices} \to \mathbf{H}^2$$

$$\rho_0 : \pi_1(M) \to Isom(\mathbf{H}^2) \subset Isom^+(\mathbf{H}^3) \cong PSL_2(\mathbf{C})$$
$$Dev_0(\gamma x) = \rho_0(\gamma) Dev_0(x)$$

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1) Deform ρ_0 to ρ_t (with $\rho_t(\text{meridian}) = \text{rotation}$) for $t \geq 0$

2) Find ρ_t-equivariant immersions $Dev_t : \widetilde{M} \to \mathbf{H}^3$ for $t > 0$ and control them around $Sing$

Deforming representations

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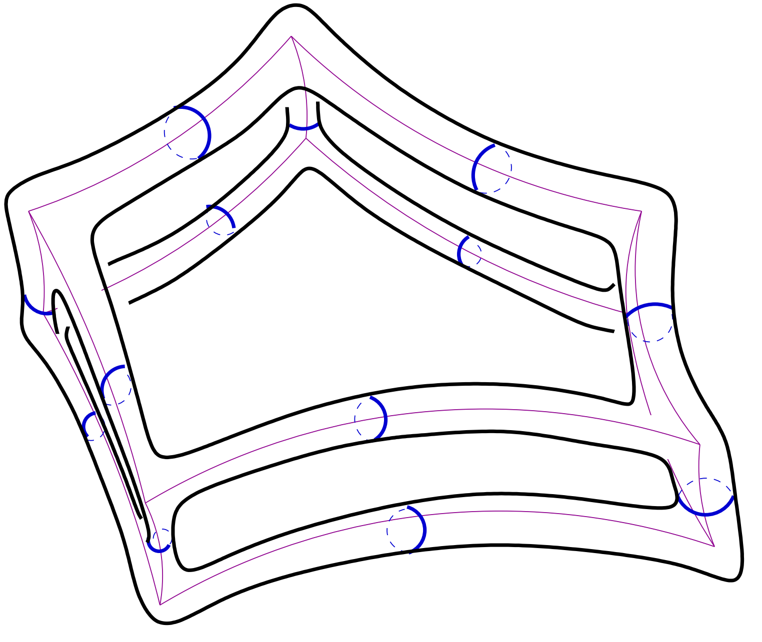

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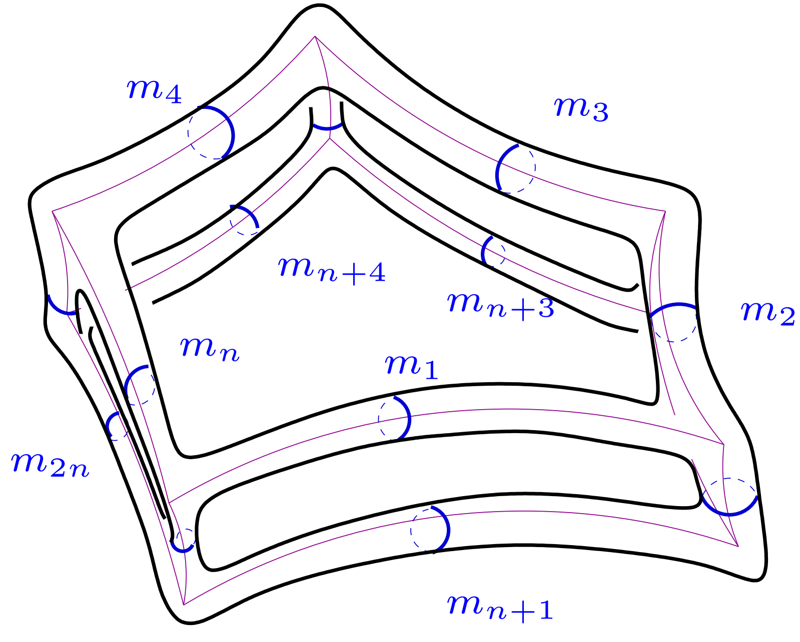

$m_1, \ldots, m_n$ horizontal

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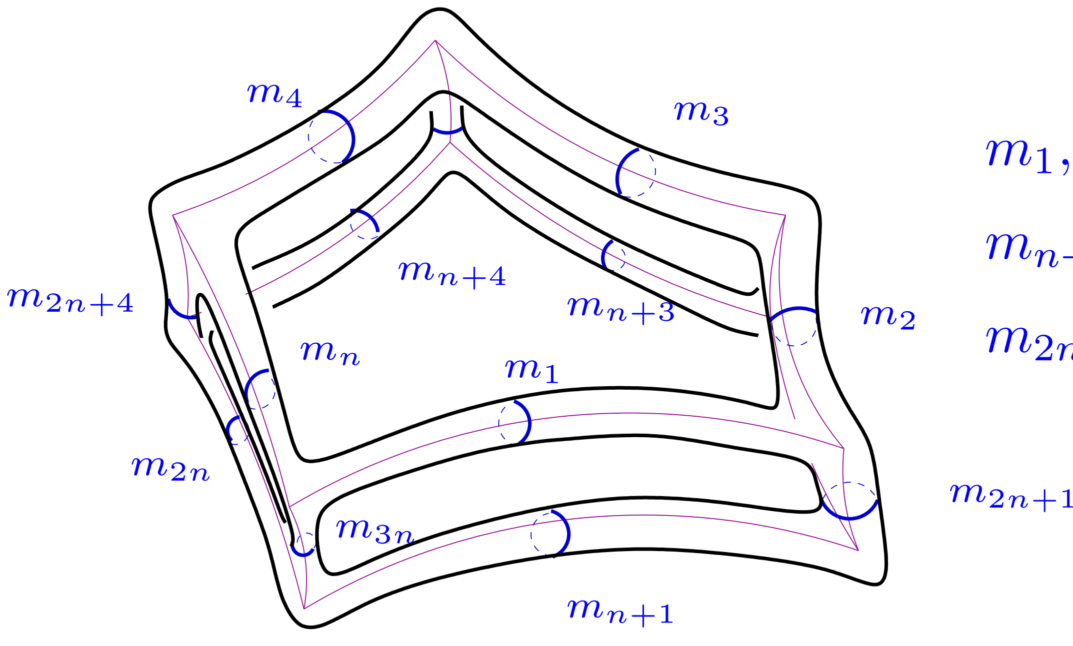

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$m_{2n+1}, \ldots, m_{3n}$	vertical

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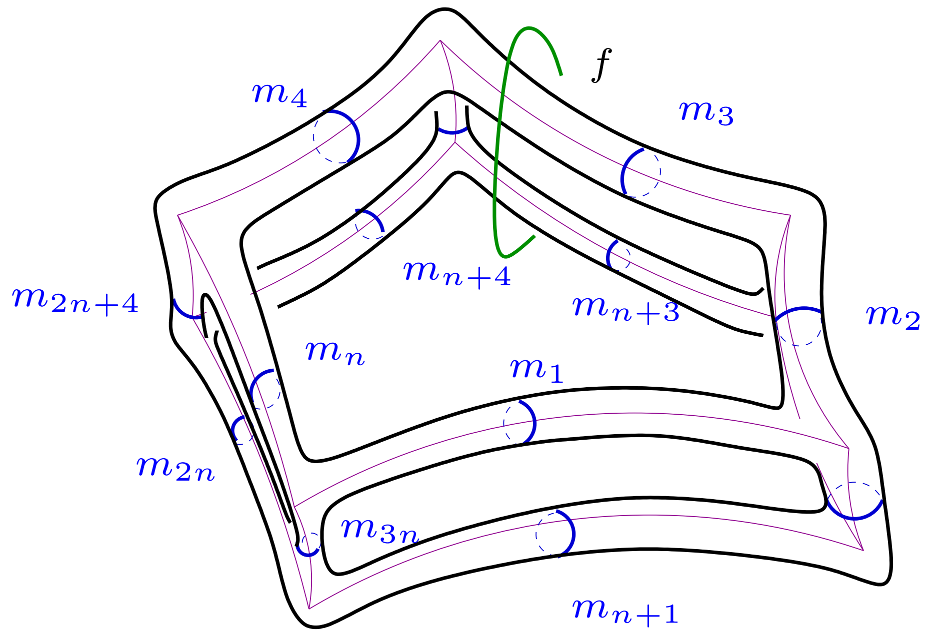

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$m_{2n+1}, \ldots, m_{3n}$	vertical

Want to find $\rho_t : \pi_1(M) \to PSL_2(\mathbf{C})$, $t \geq 0$ such that

1) $\rho_t(m_1), \ldots, \rho_t(m_{2n})$ rotation of angle $\pi - t$

2) $\rho_t(m_{2n+1}), \ldots, \rho_t(m_{3n})$ rotation of angle $\beta_1, \ldots, \beta_n$ resp.

3) $\rho_t(f)$ loxodromic with axis almost perpendicular to $\mathbf{H}^2$

Variety of characters of a surface

- ∂M surface of genus $n+1$

$$X(\partial M) = \hom(\pi_1(\partial M), PSL_2(\mathbf{C}))//PSL_2(\mathbf{C})$$

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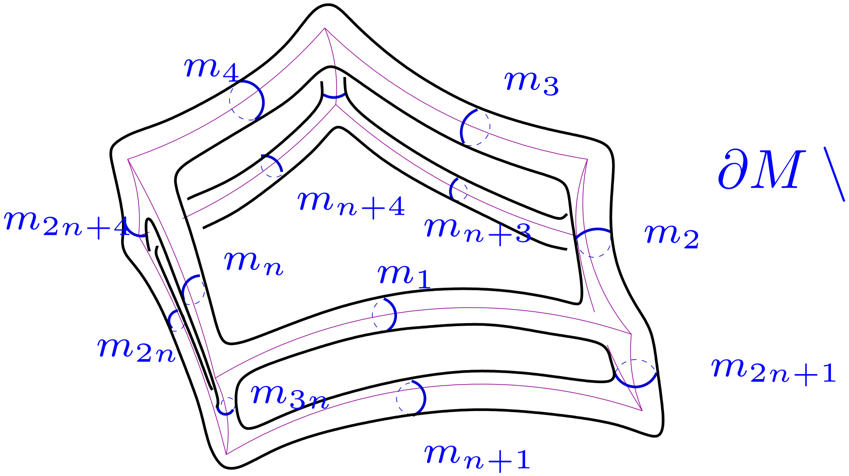

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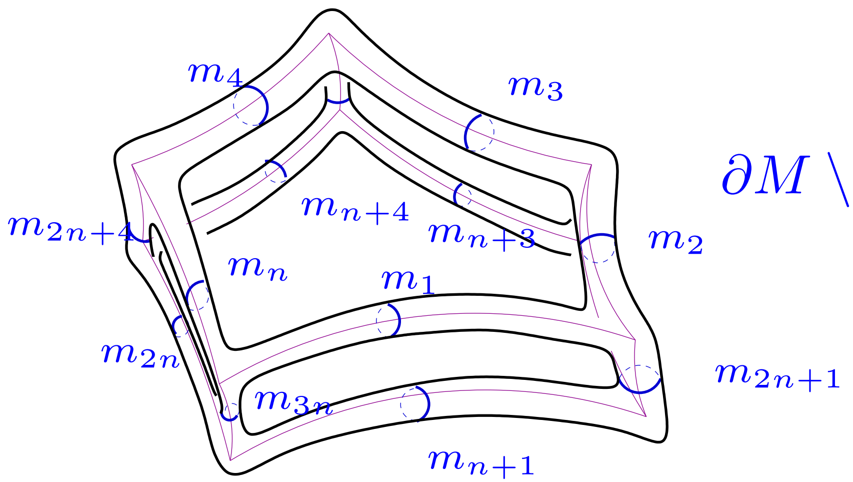

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- μ_i = complex length of m_i, $\rho(m_i) \sim \pm \begin{pmatrix} e^{\mu_i/2} & 0 \\ 0 & e^{-\mu_i/2} \end{pmatrix}$

  $\rho(m_j)$ = rotation angle $\alpha \Leftrightarrow \mu_j = \alpha i$
- λ_i = complex length conj. amalgam (= length of the i-th edge)

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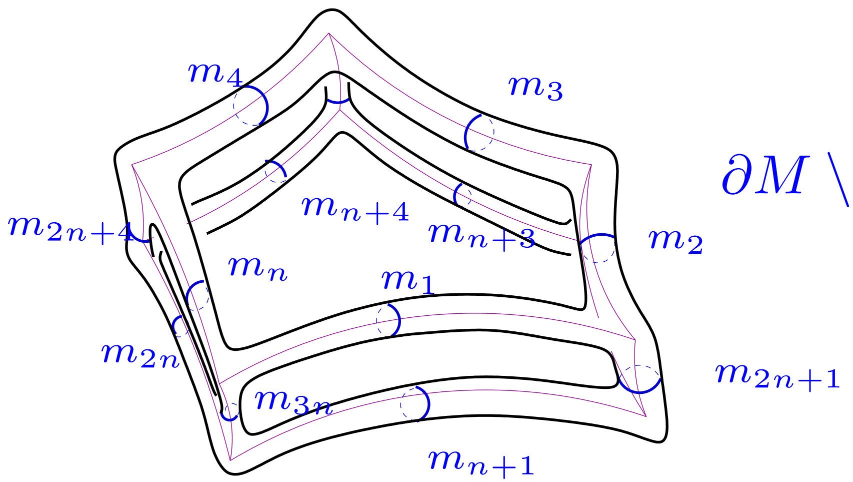

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  Complex Fenchel-Nielsen:

$$(\mu_1, \ldots \mu_{3n}, \lambda_1, \ldots \lambda_{3n}) : U \subset X(\partial M) \to \mathbf{C}^{6n} \text{ local coordinates}$$

Finding an infinitesimal regeneration

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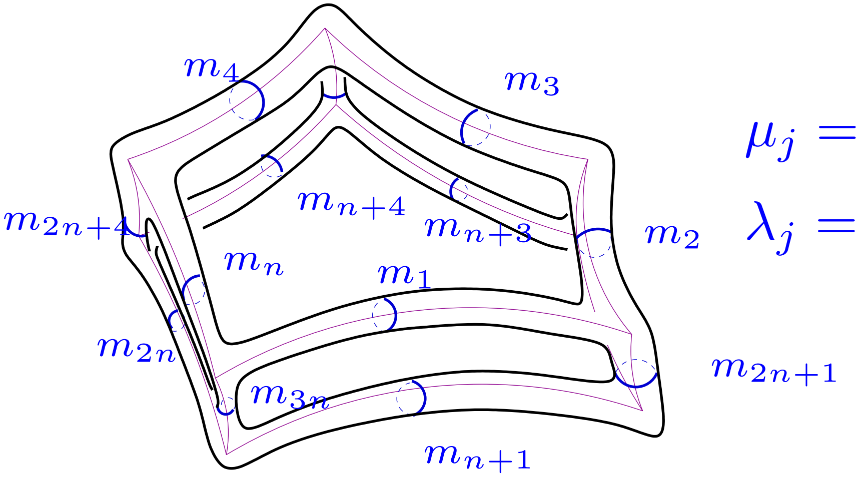

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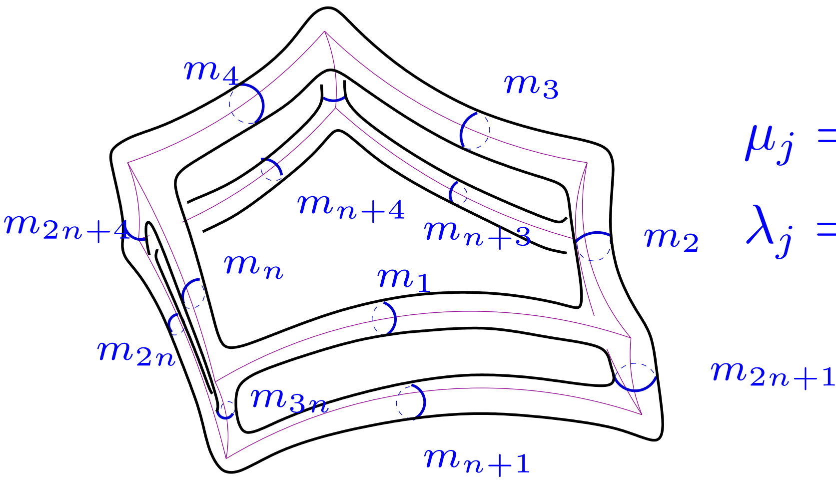

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Lemma (infinitesimal regeneration)

ρ_0 is a critical point of $\lambda_1 + \cdots \lambda_{2n}$ in $X(M) \cap \{\mu_i = ctnt\}$"$\cong X(\mathcal{P}^2)$"

$\Rightarrow$ there exists $v \in T_{\rho_0}X(M)$, $\begin{cases} d\mu_j(v) = 1 \text{ if } m_j \text{ horiz. } (j \leq 2n) \\ d\mu_j(v) = 0 \text{ if } m_j \text{ vert. } (j \geq 2n+1) \end{cases}$

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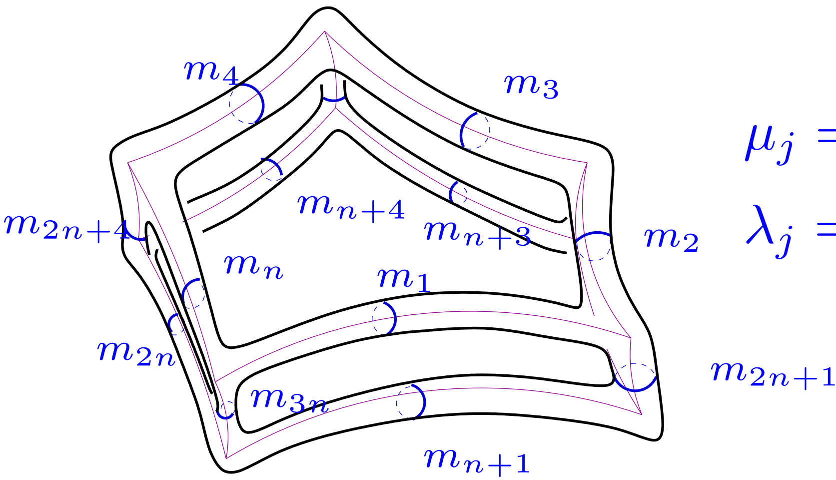

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"v comes from the Hamiltonian vector field of the perimeter in $X(\partial M)$"

Goldman's symplectic structure

- Weil: $T_\rho X(\partial M) \cong H^1(\partial M; \mathfrak{sl}_2(\mathbf{C})_{Ad\rho})$ (also true for M)

$$\omega : T_\rho X(\partial M) \times T_\rho X(\partial M) \overset{\cup}{\to} H^2(\partial M, \mathfrak{sl}_2 \times \mathfrak{sl}_2) \xrightarrow{\text{Killing}} H^2(\partial M; \mathbf{C}) \cong \mathbf{C}$$

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- $res : X(M) \to X(\partial M)$ restriction map

Hodgson: $res_*(T_{\rho_0} X(M)) \subset T_{\rho_0} X(\partial M)$ is a Lagrangian subspace

($\omega|_{res_*(T_{\rho_0} X(M))} = 0$ and $\mathrm{rank}(res_*) = \frac{1}{2}\dim(T_{\rho_0} X(\partial M))$)

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- Goldman: ∂_{λ_j} is the Hamiltonian vector field of μ_j:

$$\omega(\partial_{\lambda_j}, \cdot) = d\mu_j$$

$(\mu_i, \lambda_i) : U \subset X(\partial M) \to \mathbf{C}^{6n}$ local coordinates

ρ_0 is a critical point of $\lambda_1 + \cdots \lambda_{2n}$ in $X(M) \cap \{\mu_i = ctnt\}$"$\cong X(\mathcal{P}^2)$"

$\Rightarrow$ there exists $v \in T_{\rho_0}X(M)$, $\begin{cases} d\mu_j(v) = 1 \text{ if } m_j \text{ horiz. } (j \leq 2n) \\ d\mu_j(v) = 0 \text{ if } m_j \text{ vert. } (j \geq 2n+1) \end{cases}$

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- Proof.

$$res^*(d(\lambda_1 + \cdots + \lambda_{2n})) = \sum_{j=1}^{3n} a_j res^*(d\mu_j).$$

$$F = \lambda_1 + \cdots + \lambda_{2n} - \sum_{j=1}^{3n} a_j \mu_j, \qquad res^*(dF) = 0.$$

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$$F = \lambda_1 + \cdots + \lambda_{2n} - \sum_{j=1}^{3n} a_j \mu_j, \qquad res^*(dF) = 0.$$

Let $X_F \in T_{\rho_0} X(\partial M)$, $\omega(X_F, \cdot) = dF$

$\omega(X_F, res_* T_{\rho_0} X(M)) = dF \circ res_* = 0 \Rightarrow$ $X_F \in res_* T_{\rho_0} X(M)$

Let $v \in T_{\rho_0} X(M)$ satisfy $res_*(v) = -X_F$

ρ_0 is a critical point of $\lambda_1 + \cdots \lambda_{2n}$ in $X(M) \cap \{\mu_i = ctnt\}$"$\cong X(\mathcal{P}^2)$"

$\Rightarrow$ there exists $v \in T_{\rho_0}X(M)$, $\begin{cases} d\mu_j(v) = 1 \text{ if } m_j \text{ horiz. } (j \leq 2n) \\ d\mu_j(v) = 0 \text{ if } m_j \text{ vert. } (j \geq 2n+1) \end{cases}$

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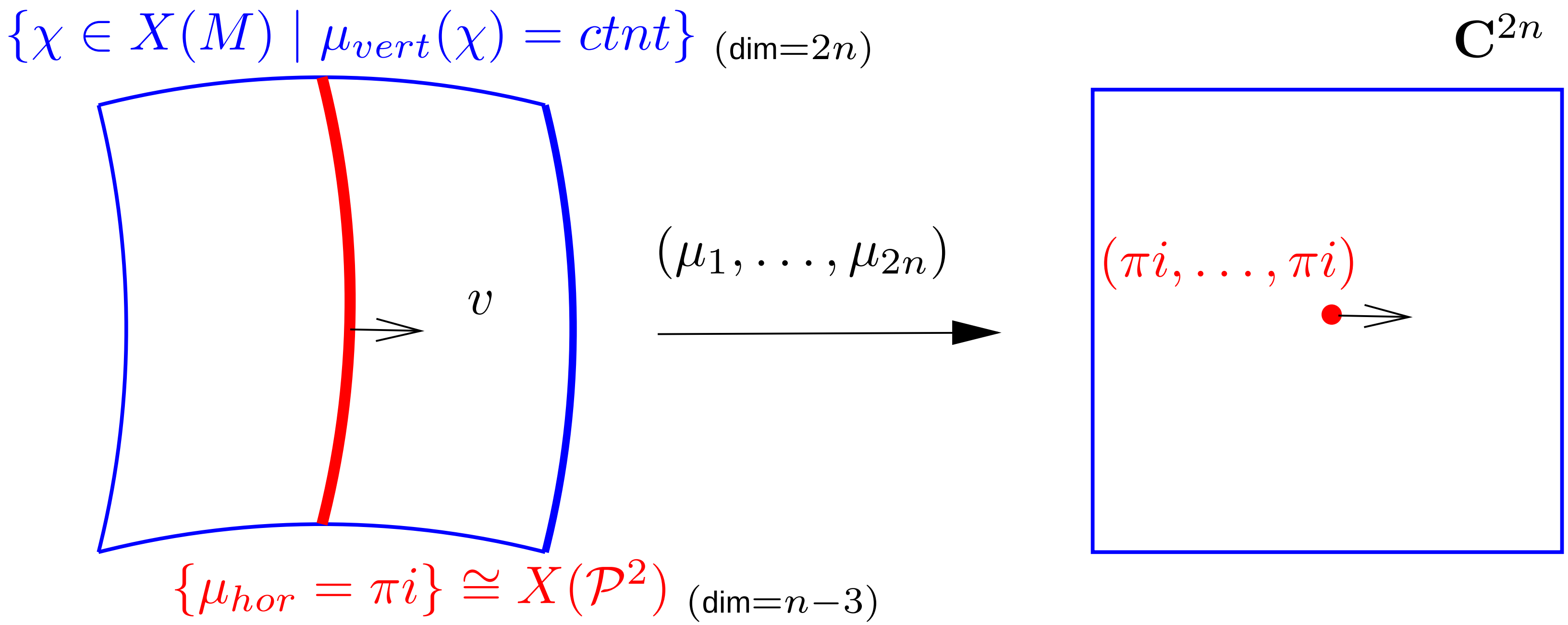
$$d\mu_j(v) = d\mu_j(-X_F) = \omega(\partial_{\lambda_j}, -X_F) = \omega(X_F, \partial_{\lambda_j}) = \frac{\partial F}{\partial \lambda_j} = \begin{cases} 1 \text{ if } \mu_j \text{ horiz.} \\ 0 \text{ if } \mu_j \text{ vert.} \end{cases}$$

The map $(\mu_1, \ldots, \mu_{2n})$ ***of horizontal*** μ***'s***



- $\{\chi \in X(M) \mid \mu_{vert}(\chi) = cnt\}$ is smooth at χ_{ρ_0}, of dim $2n$

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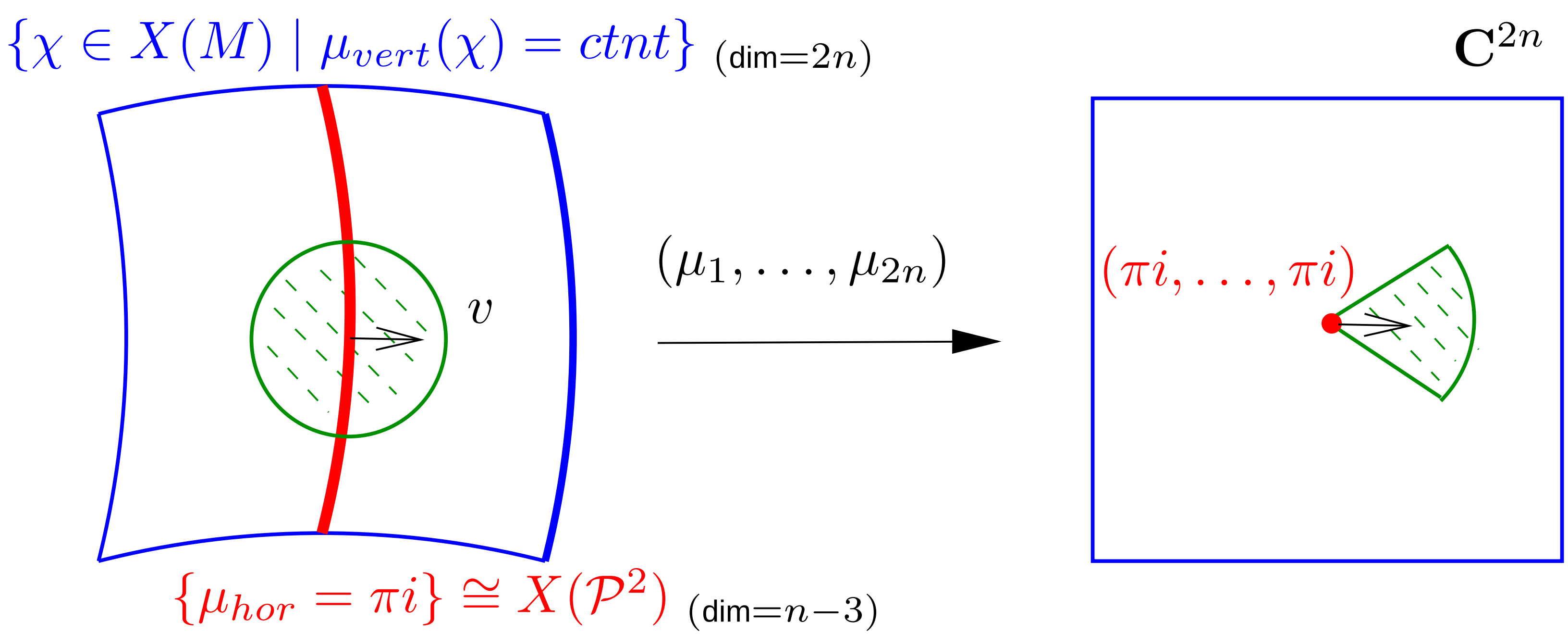
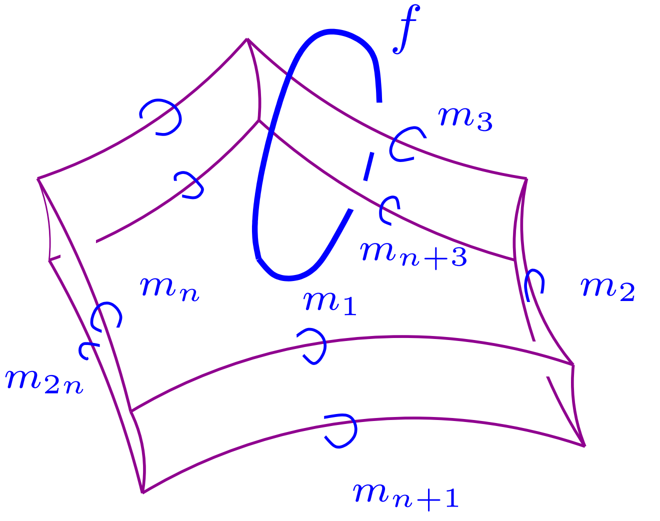


- $\{\chi \in X(M) \mid \mu_{vert}(\chi) = cnt\}$ is smooth at χ_{ρ_0}, of dim $2n$

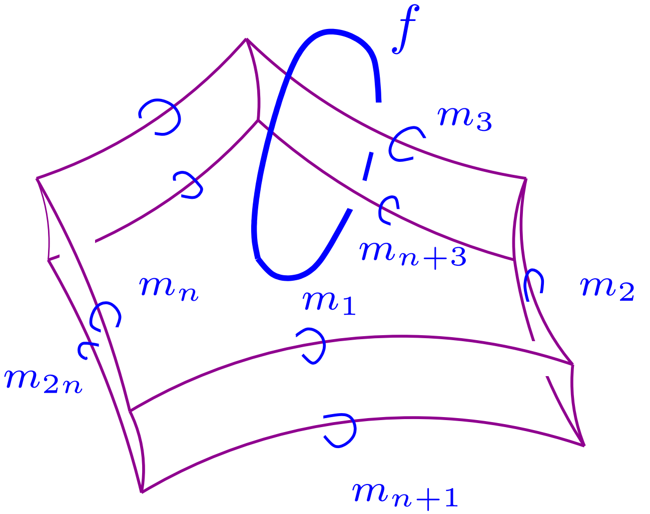
- blow up at $\{\mu_{hor} = \pi i\}$ on the left, and at $(\pi i, \dots, \pi i)$ on the right, get a locally bianalytic map between blown up mflds.

  (The Hessian of the perimeter on $X(\mathcal{P}^2)$ is nonzero).

The fibre

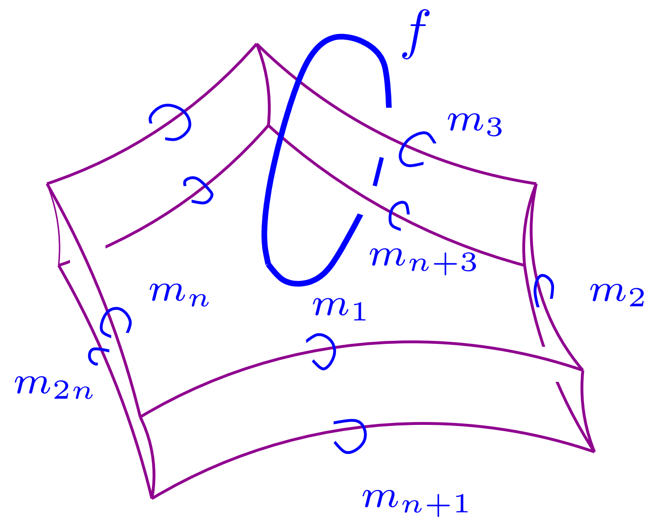


- $f = m_1 m_{n+1} = m_2 m_{n+2} = \cdots = m_n m_{2n} \in \pi_1(M)$



- $f = m_1 m_{n+1} = m_2 m_{n+2} = \cdots = m_n m_{2n} \in \pi_1(M)$
  $vol(\rho_t) \geq c\,t$, $c > 0 \Rightarrow \rho_t(f)$ nontrivial for $t > 0$.

$$\rho_t(f) = \exp(t\mathfrak{a} + O(t^2)) \qquad \mathfrak{a} \in \mathfrak{sl}_2(\mathbf{C}),\ \mathfrak{a} \neq 0$$



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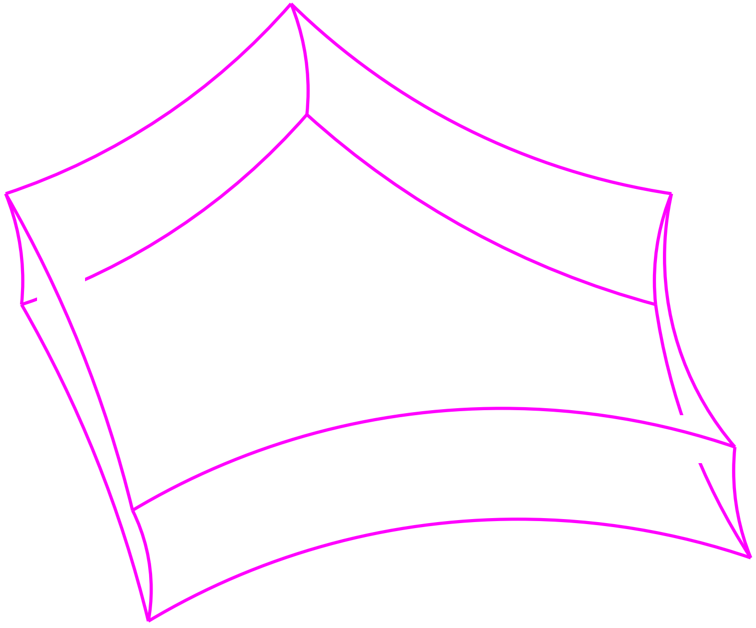
$$f = m_j m_{n+j} \Rightarrow \begin{cases} \mathfrak{a} \text{ is an infinitesimal pure loxodromic element, and} \\ d_{\mathbf{C}}(\text{axis}(\mathfrak{a}), \text{axis}(\rho_0(m_j))) \text{ is independent of } j \\ \Rightarrow \ \text{axis}(\mathfrak{a}) \text{ is perpendicular to } \mathbf{H}^2 \end{cases}$$

Once we have ρ_t define $Dev_t : \tilde{M} \rightarrow \mathbf{H}^3$



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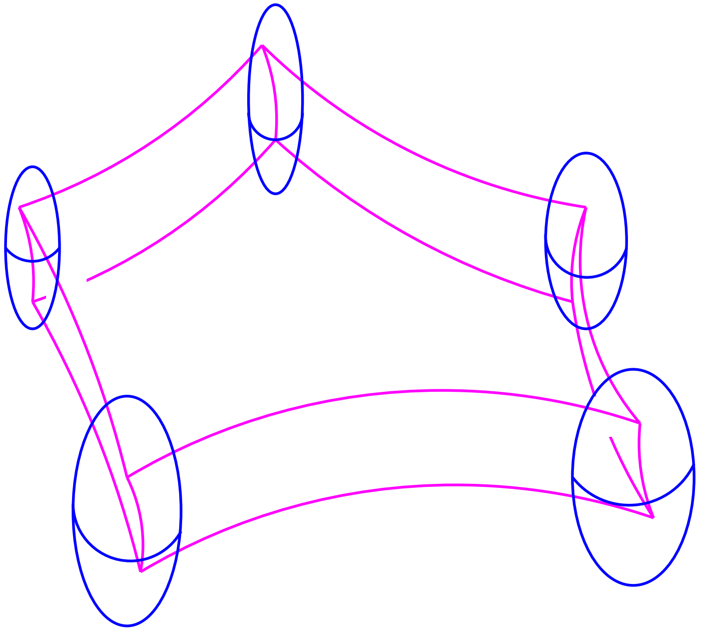

1st) on a neighborhood of the singular fibres

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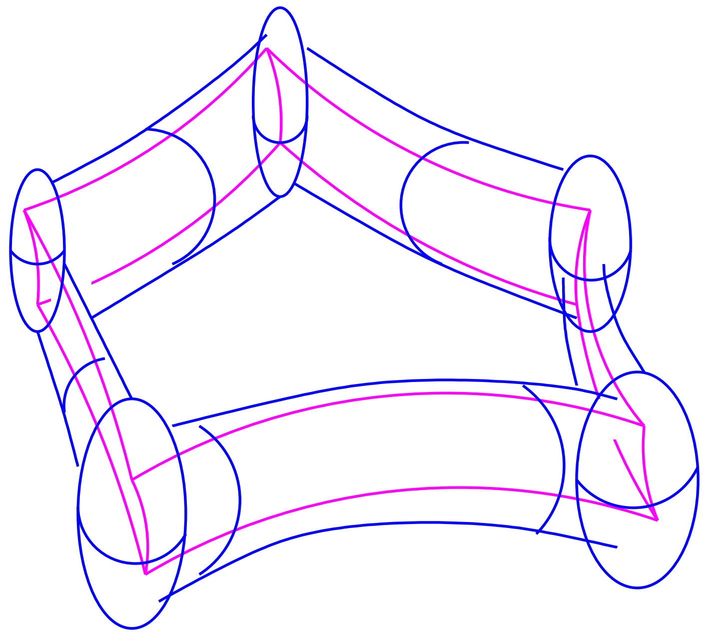
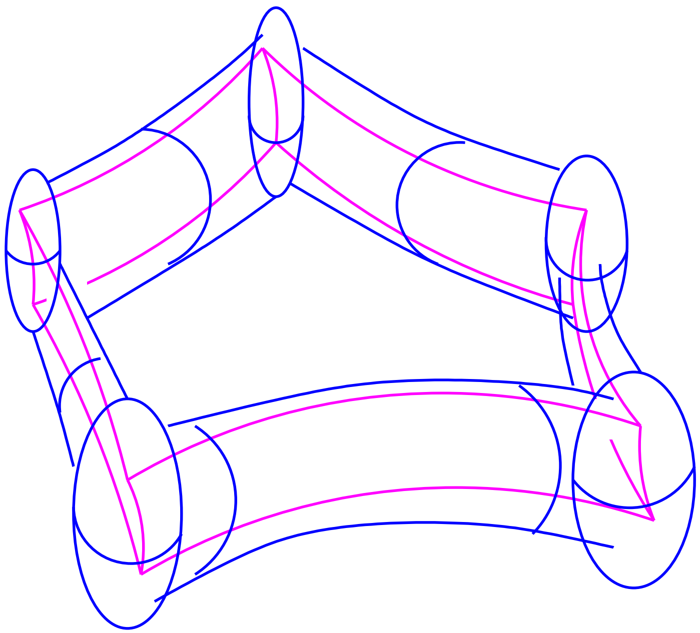


1st) on a neighborhood of the singular fibres

2nd) on a neigborhood of the singular edges

Developing maps

Once we have ρ_t define $Dev_t : \tilde{M} \to \mathbf{H}^3$



1st) on a neighborhood of the singular fibres

2nd) on a neigborhood of the singular edges

3rd) on the interior fibres of the polygon: solid torus in $\mathbf{H}^3/\rho_t(f)$