Singularities of free group character varieties

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- Any complex affine group G which arises in this fashion is called *reductive*.

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- The conjugation action of G on $\mathfrak{R}_r(G)$ is regular; that is, $G \times \mathfrak{R}_r(G) \to \mathfrak{R}_r(G)$ is given by polynomials.
- In particular, the action is $(g, \rho) \mapsto g\rho g^{-1}$ or equivalently $(g, (\rho(\mathtt{x}_1), ..., \rho(\mathtt{x}_r))) \mapsto \left(g\rho(\mathtt{x}_1)g^{-1}, ..., g\rho(\mathtt{x}_r)g^{-1}\right)$.

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- In both cases we consider the topology induced by an ambient affine space of minimal dimension.

A Topology Theorem

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- If Γ is the fundamental group of a closed surface and $G = \mathrm{SL}(2, \mathbb{C})$ Wentworth, Daskalopoulos, and Wilkin (2008) show $\mathsf{P}_t\left(\mathfrak{X}_\Gamma(G)\right) = \mathsf{P}_t\left(\mathfrak{X}_\Gamma(K)\right) + C_\Gamma(t)$, where $C_\Gamma \neq 0$.

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- If Γ is free abelian and $G = \mathrm{GL}(n, \mathbb{C})$ the answer is affirmative.
- In "The topology of the moduli space of G-valued quivers" we obtain group theoretic conditions on Γ to ensure $\mathfrak{X}_\Gamma(G)$ and $\mathfrak{X}_\Gamma(K)$ are homotopy equivalent.

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$$\mathsf{P}_t(\mathfrak{X}_r) = 1 + t - \frac{t(1+t^3)^r}{1-t^4} + \frac{t^3}{2}\left(\frac{(1+t)^r}{1-t^2} - \frac{(1-t)^r}{1+t^2}\right).$$

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Proof. In 2008 T. Baird established using methods of equivariant cohomology (in his PhD thesis) that the Poincaré polynomial for $\mathfrak{X}_r(\mathrm{SU}(2)) \cong \mathrm{SU}(2)^{\times r}/\mathrm{SU}(2)$ is P_t above. □

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Observe that this means $\mathsf{P}_t(\mathfrak{X}_1) = 1 = \mathsf{P}_t(\mathfrak{X}_2)$ and $\mathsf{P}_t(\mathfrak{X}_3) = 1 + t^6$.

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Proof. Since K is assumed to be connected and simply connected, $\mathfrak{R}_r(K) \cong K^r$ is simply connected as well. Bredon has shown that a path connected K-space X has the property that $X \to X/K$ induces a surjection on fundamental groups. We conclude that $\mathfrak{X}_r(K) = \mathfrak{R}_r(K)/K$ is simply connected. By Theorem 0.1, $\mathfrak{X}_r(K_{\mathbb{C}})$ is likewise simply connected. □

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In particular, $\mathfrak{X}_r(\mathrm{SL}(n, \mathbb{C}))$ and $\mathfrak{X}_r(\mathrm{SU}(n))$ are simply connected.

Another cool corollary

Corollary 0.4. *For* $m \geq 2$ *or* $m = 0$,

$$\pi_m(\mathfrak{X}_r(\mathrm{GL}(n,\mathbb{C}))) \cong \pi_m(\mathfrak{X}_r(\mathrm{U}(n))) \cong \pi_m(\mathfrak{X}_r(\mathrm{SU}(n))) \cong \pi_m(\mathfrak{X}_r(\mathrm{SL}(n,\mathbb{C}))),$$

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Proof.

$$\mathfrak{X}_r(\mathrm{SU}(n)) \to \mathfrak{X}_r(\mathrm{U}(n)) \overset{\det}{\to} \left(S^1\right)^r := T_r.$$

is a fibration. We compute the long exact homotopy sequence:

$$\cdots \to \pi_m(\mathfrak{X}_r(\mathrm{SU}(n))) \to \pi_m(\mathfrak{X}_r(\mathrm{U}(n))) \to \pi_m(T_r) \to \cdots \to \pi_0(T_r) \to 1.$$

Using the fact that S^1 has a contractible universal cover which implies $\pi_m(T_r) = 1$ for $m \geq 2$, one calculates in these cases $\pi_m(\mathfrak{X}_r(\mathrm{U}(n))) \cong \pi_m(\mathfrak{X}_r(\mathrm{SU}(n)))$. □

Singularities

Lemma 0.5 (Singular Equivalence)**.** *Let* $[\rho] \in \mathfrak{X}_r(\mathrm{SL}(n, \mathbb{C}))$ *and let* $[\psi] \in \mathfrak{X}_r(\mathrm{SU}(n))$*. Then*

1. $[\rho] \in \mathfrak{X}_r(\mathrm{SL}(n, \mathbb{C}))^{sing}$ *if and only if* $[\rho] \in \mathfrak{X}_r(\mathrm{GL}(n, \mathbb{C}))^{sing}$

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Proof. First let $[\rho] \in \mathfrak{X}_r(\mathrm{SL}(n,\mathbb{C}))$. One can show that central multiplication mapping $\mathfrak{X}_r(\mathrm{SL}(n,\mathbb{C})) \times (\mathbb{C}^*)^r \to \mathfrak{X}_r(\mathrm{GL}(n,\mathbb{C}))$ is an étale equivalence and such mappings preserve tangent spaces, we conclude

$$T_{[\rho]}(\mathfrak{X}_r(\mathrm{GL}(n,\mathbb{C}))) \cong T_{[\rho]}(\mathfrak{X}_r(\mathrm{SL}(n,\mathbb{C})) \times (\mathbb{C}^*)^r) \cong T_{[\rho]}(\mathfrak{X}_r(\mathrm{SL}(n,\mathbb{C}))) \oplus \mathbb{C}^r.$$

□

Main Theorem

- For $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{C})$ irreducible representations do not admit any proper (non-trivial) invariant subspaces.
- Denote the set of reducible representations by $\mathfrak{X}_r(G)^{red}$.

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Theorem 0.6. *Let* $r,n \geq 2$*. Let* G *be* $\mathrm{SL}(n,\mathbb{C})$ *or* $\mathrm{GL}(n,\mathbb{C})$ *and* K *be* $\mathrm{SU}(n)$ *or* $\mathrm{U}(n)$*. Then* $\mathfrak{X}_r(G)^{red} = \mathfrak{X}_r(G)^{sing}$ *and* $\mathfrak{X}_r(K)^{red} = \mathfrak{X}_r(K)^{sing}$ *if and only if* $(r,n) \neq (2,2)$*.*

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Theorem 0.6. *Let $r,n \geq 2$. Let G be $\mathrm{SL}(n,\mathbb{C})$ or $\mathrm{GL}(n,\mathbb{C})$ and K be $\mathrm{SU}(n)$ or $\mathrm{U}(n)$. Then $\mathfrak{X}_r(G)^{red} = \mathfrak{X}_r(G)^{sing}$ and $\mathfrak{X}_r(K)^{red} = \mathfrak{X}_r(K)^{sing}$ if and only if $(r,n) \neq (2,2)$.*

Heusener and Porti, studying $\mathsf{PSL}(2,\mathbb{C})$, proved some of this theorem for $n=2$; their work was motivational. Also, a new paper of Sikora addresses some of the tools we use to prove this theorem in a new paper titled "Character Varieties". Lastly, similar results with respect to $\mathfrak{gl}(n,\mathbb{C})^r /\!\!/ \mathrm{GL}(n,\mathbb{C})$ have been proved by LeBruyn.

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2. $\mathfrak{X}_r(K)^{red} = \mathfrak{X}_r(K)^{sing}$ if and only if $\mathfrak{X}_r(K_{\mathbb{C}})^{red} = \mathfrak{X}_r(K_{\mathbb{C}})^{sing}$.

   Proof.

   (a) $\mathfrak{X}_r(K) \subset \mathfrak{X}_r(K_{\mathbb{C}})$

   (b) K-reducible K-representations are $K_{\mathbb{C}}$-reducible (obvious)

   (c) $K_{\mathbb{C}}$-reducible K-representations are K-reducible (not obvious)

   (d) $\dim_{\mathbb{R}} T_{[\rho]}\mathfrak{X}_r(K) = \dim_{\mathbb{C}} T_{[\rho]}\mathfrak{X}_r(K_{\mathbb{C}})$ (by definition)

   (e) $\dim_{\mathbb{R}} \mathfrak{X}_r(K) = \dim_{\mathbb{C}} \mathfrak{X}_r(K_{\mathbb{C}})$

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$$\{[\rho_1 \oplus \rho_2] \in \mathfrak{X}_r(\mathrm{GL}(n,\mathbb{C})) : \ \rho_i : \mathtt{F}_r \to \mathrm{GL}(n_i,\mathbb{C}) \text{ are irreducible}\},$$

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(c) Suppose there exists a set $\mathcal{O} \subset \mathfrak{X}_r(G)^{sing} \cap \mathfrak{X}_r(G)^{red}$ that is dense with respect to the ball topology in $\mathfrak{X}_r(G)^{red}$. Then $\mathfrak{X}_r(G)^{sing} = \mathfrak{X}_r(G)^{red}$.

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4. The conjugation action being free and proper on the irreducibles implies $\mathfrak{X}_r(G)^{sing} \subset \mathfrak{X}_r(G)^{red}$.

5. The Luna Slice Theorem (there exists $V_x \subset X$ so $(G \times V_x)/\!\!/\mathsf{Stab}_x \to X$ is strongly étale) implies

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where ρ^{ss} is a poly-stable representative from the extended orbit $[\rho]$.

6. (a) Let $\rho \in U_{r,n} \subset \mathfrak{R}_r(G)^{red}$ be of reduced type $[n_1, n_2]$ with $n_1, n_2 > 0$ and $n = n_1 + n_2$ and write it in the form $\rho = \rho_1 \oplus \rho_2 = \begin{pmatrix} \vec{X} & \vec{0}_{n_1 \times n_2} \\ \vec{0}_{n_2 \times n_1} & \vec{Y} \end{pmatrix}$, where $\vec{X} = (X_1, ..., X_r) \in M^r_{n_1 \times n_1}$ and $\vec{Y} = (Y_1, ..., Y_r) \in M^r_{n_2 \times n_2}$ and $\vec{0}_{k \times l} = \underbrace{(0_{k \times l}, ..., 0_{k \times l})}_{r}$ where $0_{k \times l}$ is the k by l matrix of zeros. Recall that these representations form a dense set in $\mathfrak{X}_r(G)^{red}$.

(b) Let $\mathrm{diag}(a_1, ..., a_n)$ be an $n \times n$ matrix whose (i, j)-entry is 0 if $i \neq j$ and is equal to a_i otherwise. Then $\mathsf{Stab}_\rho = \mathbb{C}^* \times \mathbb{C}^*$ is given by

$$\mathrm{diag}(\underbrace{\lambda, ..., \lambda}_{n_1}, \overbrace{\mu, ..., \mu}^{n_2}).$$

We note that the action of the center is trivial so the action of the stabilizer modulo its center is $\mathbb{C}^*$.

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(c) Then the cocycles satisfy

$$Z^1(\mathbb{F}_r; \mathrm{Ad}_\rho) \cong \mathfrak{g}^r =$$

$$= \left\{ \begin{pmatrix} \vec{A} & \vec{B} \\ \vec{C} & \vec{D} \end{pmatrix} \;\middle|\; \vec{A} \in M^r_{n_1 \times n_1}, \vec{B} \in M^r_{n_1 \times n_2}, \vec{C} \in M^r_{n_2 \times n_1}, \vec{D} \in M^r_{n_2 \times n_2} \right\},$$

which have dimension $n^2 r$ since this is the tangent space to a representation and the representation variety is smooth.

(d) The coboundaries are given by $B^1(\mathtt{F}_r; \mathrm{Ad}_\rho) \cong$

$$\cong \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} \vec{X} & \vec{0}_{n_1 \times n_2} \\ \vec{0}_{n_2 \times n_1} & \vec{Y} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \vec{X}^{-1} & \vec{0}_{n_1 \times n_2} \\ \vec{0}_{n_2 \times n_1} & \vec{Y}^{-1} \end{pmatrix} \right\}$$

$$\cong \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} \vec{X} A \vec{X}^{-1} & \vec{X} B \vec{Y}^{-1} \\ \vec{Y} C \vec{X}^{-1} & \vec{Y} D \vec{Y}^{-1} \end{pmatrix} \right\},$$ for a fixed element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{g}$.

It has dimension $n^2 - 2$ since it is the tangent space to the G-orbit of ρ which has dimension equal to that of the group minus its stabilizer.

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It has dimension n^2-2 since it is the tangent space to the G-orbit of ρ which has dimension equal to that of the group minus its stabilizer.

(e) Thus with respect to the torus action of the stabilizer,

$$H^1(\mathtt{F}_r;\mathrm{Ad}_\rho) \cong H^1(\mathtt{F}_r;\mathrm{Ad}_{\rho_1}) \oplus H^1(\mathtt{F}_r;\mathrm{Ad}_{\rho_2}) \oplus W,$$

where W exist since the torus action is reductive.

(f) Computing dimensions we find:

$$\begin{aligned}\dim_{\mathbb{C}} H^1(\mathtt{F}_r;\mathrm{Ad}_\rho) &= n^2r-(n^2-2)=n^2(r-1)+2,\\ \dim_{\mathbb{C}} H^1(\mathtt{F}_r;\mathrm{Ad}_{\rho_i}) &= n_i^2r-(n_i^2-1)=n_i^2(r-1)+1,\end{aligned}$$

which implies
$\dim_{\mathbb{C}} H^1(\mathtt{F}_r;\mathrm{Ad}_\rho)/\!\!/(\mathbb{C}^*\times\mathbb{C}^*)=n^2(r-1)+1=\dim_{\mathbb{C}}\mathfrak{X}_r(G)$,
since the diagonal of the $\mathbb{C}^*\times\mathbb{C}^*$ action is the center which acts trivially. We also conclude that

$$\dim_{\mathbb{C}} W=(n^2-n_1^2-n_2^2)(r-1)=2n_1n_2(r-1).$$

(f) Computing dimensions we find:

$$\dim_{\mathbb{C}} H^1(\mathtt{F}_r; \mathrm{Ad}_\rho) = n^2 r - (n^2 - 2) = n^2(r-1) + 2,$$
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which implies
$\dim_{\mathbb{C}} H^1(\mathtt{F}_r; \mathrm{Ad}_\rho) /\!\!/ (\mathbb{C}^* \times \mathbb{C}^*) = n^2(r-1) + 1 = \dim_{\mathbb{C}} \mathfrak{X}_r(G)$, since the diagonal of the $\mathbb{C}^* \times \mathbb{C}^*$ action is the center which acts trivially. We also conclude that

$$\dim_{\mathbb{C}} W = (n^2 - n_1^2 - n_2^2)(r-1) = 2n_1 n_2 (r-1).$$

7. Explicitly, the Stab_ρ action on $H^1(\mathtt{F}_r; \mathrm{Ad}_\rho)$ is given by:

$$\mathrm{diag}(\underbrace{\lambda, ..., \lambda}_{n_1}, \overbrace{\mu, ..., \mu}^{n_2}) \cdot \left[\begin{pmatrix} \vec{A} & \vec{B} \\ \vec{C} & \vec{D} \end{pmatrix} \right] \mapsto \left[\begin{pmatrix} \vec{A} & \lambda \vec{B} \mu^{-1} \\ \mu \vec{C} \lambda^{-1} & \vec{D} \end{pmatrix} \right].$$

The action on $H^1(\mathtt{F}_r; \mathrm{Ad}_{\rho_1}) \oplus H^1(\mathtt{F}_r; \mathrm{Ad}_{\rho_2})$ is trivial (but not so on W) and we conclude

$$H^1(\mathtt{F}_r; \mathrm{Ad}_\rho) /\!\!/ (\mathbb{C}^* \times \mathbb{C}^*) \cong H^1(\mathtt{F}_r; \mathrm{Ad}_{\rho_1}) \oplus H^1(\mathtt{F}_r; \mathrm{Ad}_{\rho_2}) \oplus (W /\!\!/ (\mathbb{C}^* \times \mathbb{C}^*)).$$

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8. Let $n \geq 2$ and $T = \mathbb{C}^* \times \mathbb{C}^*$ act on a vector space $V = \mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$ as follows:

$$(\lambda, \mu) \cdot (\mathbf{z}, \mathbf{w}) = (\lambda\mu^{-1}\mathbf{z}, \mu\lambda^{-1}\mathbf{w}).$$

Then, $\mathbb{C}^{2n}/\!\!/T$ is isomorphic to the affine cone over the product of projective spaces $\mathcal{C}_{\mathbb{C}}(\mathbb{C}\mathsf{P}^{n-1} \times \mathbb{C}\mathsf{P}^{n-1})$. Its unique singularity is the orbit of the origin. This follows since the invariant polynomials are generated by $z_j w_k$.

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$$H^1(\mathtt{F}_r;\mathrm{Ad}_\rho)/\!\!/(\mathbb{C}^*\times\mathbb{C}^*)\cong H^1(\mathtt{F}_r;\mathrm{Ad}_{\rho_1})\oplus H^1(\mathtt{F}_r;\mathrm{Ad}_{\rho_2})\oplus(W/\!\!/(\mathbb{C}^*\times\mathbb{C}^*)).$$

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9. Therefore, 0 is a singularity of $W/\!\!/(\mathbb{C}^*\times\mathbb{C}^*)$ which then implies it is a singularity to $H^1(\mathtt{F}_r;\mathrm{Ad}_\rho)/\!\!/(\mathbb{C}^*\times\mathbb{C}^*)$ whenever $\dim_{\mathbb{C}}W=2n_1n_2(r-1)>2$; that is, whenever $(r,n)\neq(2,2)$. $\square$

Picture Book of Topologies (Examples)

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2. For $G = \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^* \implies K = \mathrm{U}(1) \cong S^1$. In these cases the conjugation action is trivial $\implies$

$$\mathfrak{X}_r(K) \cong (S^1)^{\times r} \text{ and } \mathfrak{X}_r(G) \cong (\mathbb{C}^*)^{\times r}.$$

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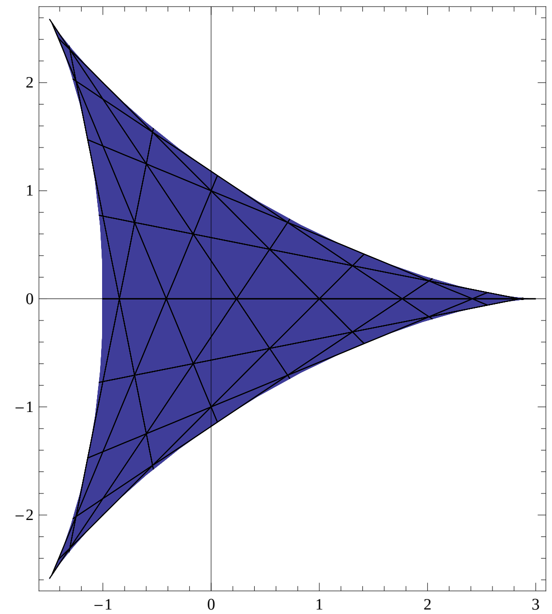

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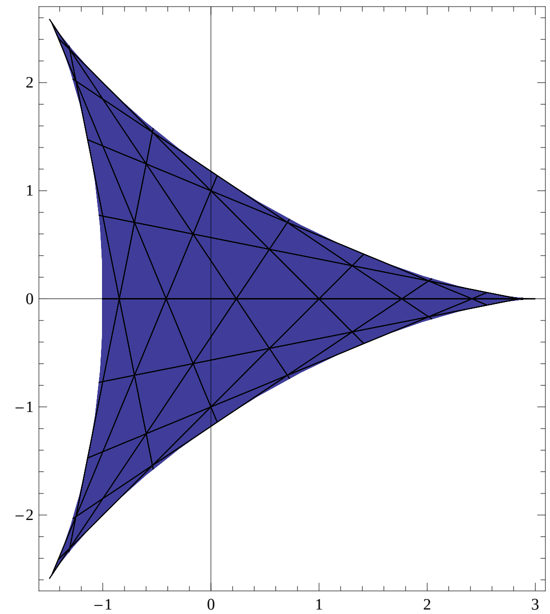

Figure 1: $\mathrm{SU}(3)/\mathrm{SU}(3)$

And also we have $\mathfrak{X}_1(\mathrm{SL}(n,\mathbb{C})) \cong \mathbb{C}^{n-1}$ given by the coefficients of the characteristic polynomial.

4. The Fricke-Vogt Theorem (1896,1889) tells that $\mathfrak{X}_2(\mathrm{SL}(2,\mathbb{C})) \cong \mathbb{C}^3$ given by

$$[\rho] \mapsto (\mathrm{tr}(X), \mathrm{tr}(Y), \mathrm{tr}(XY)),$$

where $\rho \leftrightarrow (X, Y) \in \mathrm{SL}(2,\mathbb{C})^{\times 2}$.

On the other hand, in 1992 Jeffrey and Weitsman compute that $\mathfrak{X}_2(\mathrm{SU}(2)) \cong$

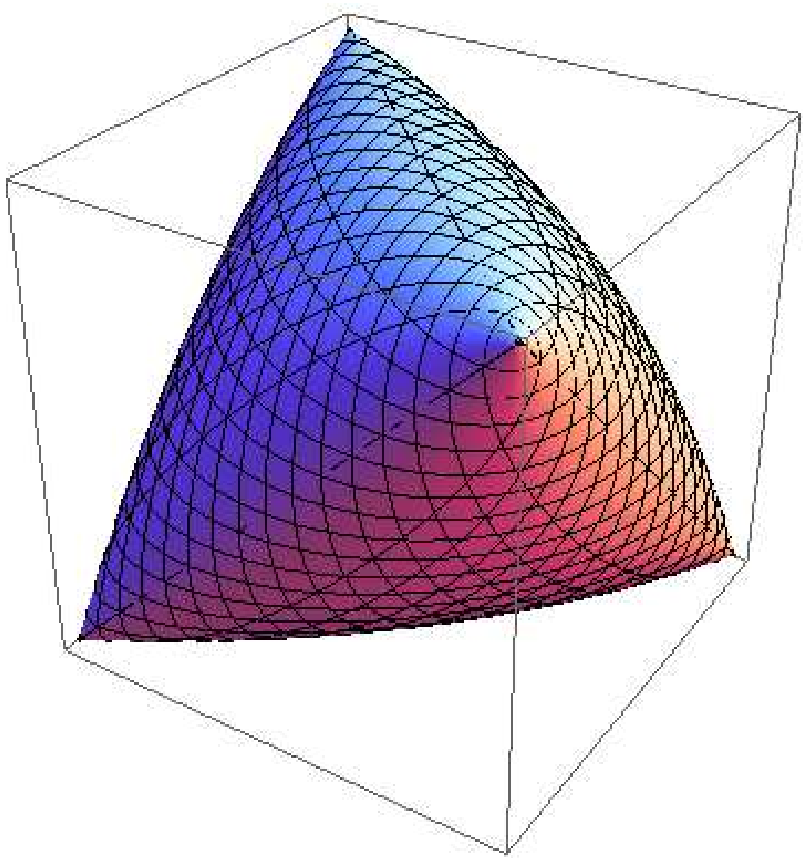

Figure 2: $\mathrm{SU}(2)^{\times 2}/\mathrm{SU}(2)$

5. Let $G = \mathrm{SL}(2, \mathbb{C})$.

$$\begin{array}{ccc} \mathfrak{X}_3(G) & \hookrightarrow & \mathbb{C}^7 \\ \downarrow & & \\ \mathbb{C}^6 & & \end{array}$$

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This statement is equivalent to the commutative algebra statement:

$$\mathbb{C}[\mathrm{Hom}(\mathtt{F}_3, \mathrm{SL}(2,\mathbb{C}))]^{\mathrm{SL}(2,\mathbb{C})} \cong \mathbb{C}[t_1, t_2, t_3, t_4, t_5, t_6][t_7]/\mathfrak{I}.$$

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- The key observations for this result are already present in 1889 by Vogt, and the related scheme $\mathfrak{gl}(2, \mathbb{C})^{\times 3} /\!\!/ \mathrm{GL}(2, \mathbb{C})$ was described by Sibirskii in 1968.

- However, an explicit description of $\mathfrak{X}_3(G)$ in the above terms seems to be appearing only now in Goldman's *Trace coordinates on Fricke spaces of some simple hyperbolic surfaces* in the Handbook of Teichmüller Theory II.

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It is worth noting that Cooper & Bratholdt motivated much of our present work.

6. In 2006, L- showed

$$\begin{array}{ccc} \mathfrak{X}_2(\mathrm{SL}(3,\mathbb{C})) & \hookrightarrow & \mathbb{C}^9 \\ \downarrow & & \\ \mathbb{C}^8 & & \end{array}$$

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Theorem 0.7 (Florentino & L-, 2008)**.** *Let* $K = \mathrm{SU}(3)$*. Then* $\mathfrak{X}_2(K_{\mathbb{C}}) \simeq S^8 \cong \mathfrak{X}_2(K)$.

Classification Theorem

Theorem 0.8. *Let* $r, n \geq 2$. *Let* G *be* $\mathrm{SL}(n, \mathbb{C})$ *or* $\mathrm{GL}(n, \mathbb{C})$ *and* K *be* $\mathrm{SU}(n)$ *or* $\mathrm{U}(n)$. $\mathfrak{X}_r(G)$ *is a topological manifold possibly with boundary if and only if* $(r, n) = (2, 2)$. $\mathfrak{X}_r(K)$ *is a topological manifold possibly with boundary if and only if* $(r, n) = (2, 2), (2, 3),$ *or* $(3, 2)$.

Thank you!

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references are available upon request