Supersymmetric Surface Operators, Four-Manifold Theory, and Invariants in Various Dimensions

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Meng-Chwan Tan
NUS

Organization of Talk

- A summary of established mathematical results that we shall proof purely physically.
- A summary of the new mathematical results as suggested by the underlying physics.
- Surface operators and singular Donaldson invariants in mathematics and physics.

Organization of Talk

- Explain the physical proofs of the established mathematical results mentioned in pt. 1.
- Explain the underlying physics which leads to the new mathematical results mentioned in pt. 2
- Conclusion

Established Results To Be Proved Physically

- Adjunction inequality for embedded surfaces of negative self-intersection in 4-manifolds (Oszvath-Szabo).

- A relation among the Seiberg-Witten invariants of certain 4-manifolds. (Oszvath-Szabo).

- SW = Gr (Taubes).

New Mathematical Results From the Underlying Physics

- Identities among the Gromov-Taubes invariants.

- Affirming a knot-homology conjecture by Kronheimer-Mrokwa.

New Mathematical Results From the Underlying Physics

- The Gromov-Taubes invariants on symplectic 4-manifold X = M x S1, and the instanton Floer homology and the Casson-Walker-Lescop invariant of M.

- The monopole Floer homology and Seiberg-Witten invariants of M.

Surface Operators

- Introduces a singularity in the SU(2) gauge field along a two-surface D in the 4-manifold.

$$A = \alpha d\theta + \cdots$$

- Nontrivial holonomy $\exp(2\pi\alpha)$ along a small circle linking D.

- Decomposition of gauge bundle along D

$$E = L \oplus L^{-1}$$

Surface Operators

- Dimension of moduli space of anti-self-dual connections:

$$s = 8k - \frac{3}{2}(\chi + \sigma) + 4l - 2(g-1)$$

- Instanton number:

$$k = -\frac{1}{8\pi^2}\int_X \mathrm{Tr}\, F \wedge F,$$

- Monopole number:

$$l = -\int_D c_1(L)$$

Singular Donaldson Invariants

- $$\mathcal{D}'_E : H_0(X\backslash D, \mathbb{R}) \oplus H_2(X\backslash D, \mathbb{R}) \to \mathbb{R}.$$
- $$\mathcal{D}'_E(p, S) = \sum_{2m+4t=s} S^m p^t d^{k'}_{m,t},$$
- $$p \in H_0(X\backslash D, \mathbb{R}) \to \Omega^0(p) \in H^4(\mathcal{M}'),$$
- $$S \in H_2(X\backslash D, \mathbb{R}) \to \Omega^2(S) \in H^2(\mathcal{M}'),$$
- $$d^{k'}_{m,t} = \int_{\mathcal{M}'} [\Omega^0(p)]^t \wedge \Omega^2(S_{i_1}) \wedge \cdots \wedge \Omega^2(S_{i_m}).$$

Singular Donaldson Invariants

- Physical interpretation via N=2 topological quantum field theory with SU(2) gauge symmetry.
- $$\langle \mathscr{O}_1 \ldots \mathscr{O}_n \rangle = \int \mathcal{D}\Phi \; \mathscr{O}_1 \ldots \mathscr{O}_n e^{-S_E},$$
- $$S_E = \frac{\{Q, V\}}{e^2} + \frac{i\Theta}{8\pi^2} \int_X \mathrm{Tr}\, F' \wedge F' - i \int_X \mathrm{Tr}\, \eta \delta_D \wedge F'$$

Singular Donaldson Invariants

- $$I_0'(p) = \frac{1}{8\pi^2}\mathrm{Tr}\langle\phi(p)\rangle^2$$
  $$I_2'(S) = -\frac{1}{\sqrt{32}\pi^2}\int_S \mathrm{Tr}\left(\langle\phi\rangle F\right)_0$$

- $$\langle [I_0'(p)]^t I_2'(S_{i_1})\dots I_2'(S_{i_m})\rangle_{k'} = \int_{\mathcal{M}'} [I_0'(p)]^t \wedge I_2'(S_{i_1}) \wedge \cdots \wedge I_2'(S_{i_m}),$$

- $$\mathbf{Z}'_{\xi,\bar{g}}(p,S) = \sum_{k'}\sum_{m\geq 0, t\geq 0}\frac{S^m}{m!}\frac{p^t}{t!}d^{k'}_{m,t} = \sum_{k'}\langle e^{pI_0'+I_2'(S)}\rangle_{k'}.$$

- $$\mathscr{D}'_\xi(S) = \sum_{k'}\left(1+\frac{1}{2}\frac{\partial}{\partial p}\right)\cdot\langle e^{pI_0'+I_2'(S)}\rangle_{k'}|_{p=0}.$$

Physics Proofs of Established Mathematical Results

- Adjunction inequality for embedded surfaces of negative self-intersection in 4-manifolds (Oszvath-Szabo).

1. $$2D \cap D - (2g-2) \leq 4l \leq (2g-2).$$

2. $$g \geq 1 \qquad l = \int_D F_L / 2\pi.$$

3. $$(2g-2) \geq D \cap D + c_1(L_d^2)[D].$$

Physics Proofs of Established Mathematical Results

4. $\Sigma \cap \Sigma \leq 0 \qquad (2g-2) \geq \Sigma \cap \Sigma - c_1(L_d^2)[\Sigma],$

5. $|c_1(L_d^2)[\Sigma]| + \Sigma \cap \Sigma \leq (2g-2).$

- Easier formula for surfaces of positive self-intersection proved by KM.
- Generalization of Thom's conjecture for CP2.

Physics Proofs of Established Mathematical Results

- A relation among the SW invariants. (Oszvath-Szabo)

1. X is a compact, oriented, symplectic four-manifold with $b_1 = 0$ and $b_2^+ > 1$.

2 $$Z'_{\mathrm{D}} = 2^{1+\frac{7\chi}{4}+\frac{11\sigma}{4}} \left\{ \sum_{\bar{\lambda}} SW_{\bar{\lambda}}\, e^{2p+S^2/2}\, e^{2(S+\tilde{\Sigma},\bar{\lambda})} + i^{(\chi+\sigma)/4} \sum_{\lambda'} SW_{\lambda'}\, (-1)^{\alpha^2\Sigma^2}\, e^{-2p-S^2/2-\tilde{\Sigma}^2}\, e^{-2i(S+i\tilde{\Sigma},\lambda)} \right\}.$$

Physics Proofs of Established Mathematical Results

3. $\alpha = \pm 1,$

$$\sum_{\widehat{\lambda}} \left\{ SW(\widehat{\lambda})\, e^{2p+S^2/2} e^{2(S,\widehat{\lambda})} + i^{(\chi+\sigma)/4}\, SW(\widehat{\lambda})\, e^{-2p-S^2/2} e^{-2i(S,\widehat{\lambda})} \right\}$$

$$= \sum_{\lambda} \left\{ SW(\lambda)\, e^{2p+S^2/2}\, e^{2(S,\lambda)} + i^{(\chi+\sigma)/4}\, SW_{\lambda'}\, e^{-2p-S^2/2}\, e^{-2i(S,\lambda)} \right\}.$$

4. $SW_{\mathfrak{s}'} = SW(\mathfrak{s})$

$\mathfrak{s}' = -i\lambda'$ and $\mathfrak{s} = -i\lambda$; $\lambda' = \lambda \mp i\delta_{\Sigma};$

$$SW_{\lambda'} = \int_{\mathcal{M}_{\mathrm{sw}}^{\lambda'}} (a_d)^{d_{L_d^2}/2}, \qquad SW(\lambda) = \sum_{x_i} (-1)^{n_i},$$

Physics Proofs of Established Mathematical Results

- Gr = SW (Taubes)

1. X is a compact, oriented, symplectic four-manifold with $b_1 = 0$ and $b_2^+ > 1$.

2. $$\sum_{p'} \mathscr{D}_0^{p'}(S) = \sum_{\bar{\lambda}} SW(\bar{\lambda})\, e^{2(S+\tilde{\Sigma},\bar{\lambda})+S^2/2+f(\chi+\sigma)},$$

3. $$\mathscr{D}_0^{k'}(S) = \mathscr{D}_0^{k'}(0) = \langle 1 \rangle_{k'}.$$

4. $$SW(\mathfrak{s}) = \langle 1 \rangle_{k'}$$

Physics Proofs of Established Mathematical Results

5. Let us now send the effective value of α to $+1$;

6. $$\langle 1 \rangle_{k'} = \sum_{x} \mathrm{sign}(\det \mathcal{D}), \qquad c_1(\mathcal{E}) = \delta_{\Sigma}.$$

7. $$SW(\widehat{\mathfrak{s}}) = \pm \mathrm{Gr}(c_1(\mathcal{E})),$$

$$\widehat{\mathfrak{s}} = \frac{1}{2} c_1(\mathscr{L}),$$

$$\mathscr{L} = K^{-1} \otimes \mathcal{E}^2.$$

New Math Results From Physics

- Relations among the Gromov-Taubes invariants.

1. X is a compact, oriented, symplectic four-manifold with $b_1 = 0$ and $b_2^+ > 1$.

2. $SW(\mathfrak{s}) = \mathrm{Gr}(c_1(\mathcal{E})),$ $\qquad \mathfrak{s} = \frac{1}{2}c_1(K^{-1} \otimes K^2)$

3. Via pt. 7 in last slide and pt. 2 above

$$\mathrm{Gr}(c_1(K)) = \pm\mathrm{Gr}(c_1(\mathcal{E})).$$

New Math Results From Physics

4. Gr(0) = 1 by definition, so from pt. 7 in 2nd last slide, we have SW(s) = 1, and so from pt. 2 above, we have

$$\mathrm{Gr}(c_1(\mathcal{E})) = +1.$$

5. Since $SW(\bar{\mathfrak{s}}) = \pm SW(-\bar{\mathfrak{s}})$, from pt. 7 in 2nd last slide, and pt. 3 above, we have

$$\mathrm{Gr}(c_1(K)) = \pm\mathrm{Gr}(c_1(K) - c_1(\mathcal{E})).$$

New Math Results From Physics

- Affirming a knot homology conjecture by KM.

1. General X = M x S1, where M = compact, closed 3-manifold and b_1(M) > 1.

2. Amenable to supersymmetric quantum mechanical interpretation. Ground states contribute to partition function.

3. Consider surface operator D = S1 x K

New Math Results From Physics

4. $\langle 1 \rangle_{K'} = \chi(HF_*(M;K;\alpha))$

5. $\chi(HF_*(M;K;\alpha)) = \sum_x \mathrm{sign}(h''(x)) = \sum_x \pm 1,$

$$\rho : \pi_1(M \backslash \tilde{K}) \to SU(2)$$

6. Can identify $HF_*(M;\tilde{K};\alpha)$ with $LI_*(M,\tilde{K})$ defined by KM

New Math Results From Physics

7. Since non-trival phase will be picked up by gauge field at knot crossings, phase is trivial along the longitude of unknot K_0

8. This implies that $LI_*(M,K)$ is zero only for unknot.

9. So, $\chi(LI_*(M,K))$ vanishes only when the symmetrized Alexander polynomial is trivial.

New Math Results From Physics

- The Gromov-Taubes invariants, instanton Floer homology, and the Casson-Walker-Lescop invariant.

1. Symplectic X = M x S1, where M = compact, closed 3-manifold and b_1(M) > 1.

2. $\alpha = +1,$ via $\langle 1 \rangle_{K'} = \chi(HF_*(M; K; \alpha))$ (pt.4 in 2nd last slide) we have

$$\mathrm{Gr}(c_1(\mathcal{E})) = \chi(HF_*(M)).$$

New Math Results From Physics

3. Agree with known mathematical results for

$X = \Sigma_g \times \mathrm{T}^2$ where $g \geq 1$

4. It has been shown that $\chi(HF_*(M)) = \lambda_{CWL}(M)$,
so we also have

$$\mathrm{Gr}(c_1(\mathcal{E})) = \lambda_{CWL}(M).$$

5. As a result, $\mathrm{Gr}(c_1(\mathcal{E})) = 0, \quad \text{if} \quad b_2^+(X) > 3.$

New Math Results From Physics

- The Instanton and monopole Floer homologies of 3-manifolds.

1. Symplectic X = M x S1, where M = compact, closed 3-manifold and b_1(M) > 1.

2. $SW(X, \pi^{-1}(\mathfrak{s}_M)) = SW(M, \mathfrak{s}_M)$, where $\pi : M \times \mathrm{S}^1 \to M$

New Math Results From Physics

3 $\chi(HM_*(M, \mathfrak{s}_M)) = SW(M, \mathfrak{s}_M)$

4. Altogether, via $\mathrm{Gr}(c_1(\mathcal{E})) = \chi(HF_*(M))$. (pt.2 in 3rd last slide) we have (up to a sign)

$$\chi(HM_*(M, \pi(\widehat{\mathfrak{s}}))) = \chi(HF_*^w(M))$$

$c_1(\widehat{\mathfrak{s}}) = c_1(\mathcal{E}) - \frac{1}{2}c_1(K)$, where K is the canonical line bundle of X, and $c_1(\mathcal{E})$ is the Poincaré-dual of the fundamental class of the *connected*, *non-multiply-covered* pseudo-holomorphic curve Σ with *positive* self-intersection.)

New Math Results From Physics

- For $M = \Sigma_g \times \mathbf{S}^1$ with $g \geq 1$, result is a generalization of a conjecture by Kronheimer for g = 0.

- For $M = \Sigma_g \times \mathbf{S}^1$ with $g \geq 1$, we have an isomorphism between $HF_*^w(M)$ <--> $QH^*(\mathcal{M}_{\Sigma_g})$, and between $HM_*(M, \pi(\widehat{\mathfrak{s}}))$ <--> $QH^*(s^r(\Sigma_g))$

- Cobordism construction smoothly linking $QH^*(\mathcal{M}_{\Sigma_g})$ and $QH^*(s^r(\Sigma_g))$

New Math Results From Physics

- An identity of the SW invariants of M.

1. Symplectic X = M x S1, where M = compact, closed 3-manifold and b_1(M) > 1.

2. Via $\mathrm{Gr}(c_1(\mathcal{E})) = \lambda_{CWL}(M).$, $SW(\hat{\mathfrak{s}}) = \pm \mathrm{Gr}(c_1(\mathcal{E})),$

   relation between Reidemeister-Milnor Torsion and $\lambda_{CWL}(M)$, and Meng-Taubes, (up to a sign)

$$SW(M, \pi(\hat{\mathfrak{s}})) = \sum_{x \in H} \sum_{\mathfrak{s}_M | \bar{c}_1(\mathfrak{s}_M) = x} SW(M, \mathfrak{s}_M)$$

New Math Results From Physics

- For $M = \Sigma_g \times S^1$ with $g \geq 1$, this is consistent with all known math results.

- Due to time constraint, we will not discuss alternative and economical physical derivations of

1. Properties of knot-homology groups from singular instantons (Kronheimer-Mrowka)

2. Vanishing theorem of monopole Floer homology of M (Kutluhan-Taubes)

3. Seiberg-Witten theory on symplectic 4-manifolds with b_1 =0 and b^+_2 > 1.

Conclusion

- Rich interplay between the physics of surface operators in supersymmetric topological gauge theory, and the geometry and topology of 4 and 3-manifolds.

- Much more can be done, especially on the physics side, which can potentially link the story to the geometric Langlands program, which involves Higgs bundles etc.