The Gardiner-Masur boundary of Teichmueller space

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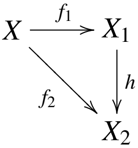
Let X be a Riemann surface of type (g, n) with $2g - 2 + n > 0$.
Let $T(X)$ be the Teichmüller space of X i.e.

$$T(X) = \{(Y, f) \mid f : X \to Y \text{ q.c.}\}/ \sim$$

where $(Y_1, f_1) \sim (Y_2, f_2)$ if there is a conformal mapping $h : Y_1 \to Y_2$ such that $h \circ f_1$ is homotopic to f_2.



Teichmüller space $T(X)$ has a canonical complete distance, called the Teichmüller distance d_T, which we recall later.

For $y = (Y, f) \in T(X)$, we set

$$Q_y = \{q \mid \text{hol. quadratic differential on } Y \text{ w. } ||q|| < \infty\},$$

where

$$||q|| = \int_Y |q| = \int_Y |q(z)| dxdy.$$

Then

$$Q = \cup_{y \in T(X)} Q_y$$

is a complex vector bundle over $T(X)$ of rank $3g - 3 + n$. Set

$$Q^1 = \{q \in Q \mid ||q|| = 1\}.$$

We set

$$\begin{aligned}\mathcal{S} &= \{\text{non-peripheral, non-trivial s.c.c.}\}/\text{isotopy} \\ \mathcal{R} &= \mathbb{R}_{\geq 0}^{\mathcal{S}} = \{\text{non-negative functions on } \mathcal{S}\} \\ \mathcal{PR} &= (\mathcal{R} - \{0\})/\mathbb{R}_+.\end{aligned}$$

We denote the projection by

$$\mathrm{proj} : \mathcal{R} - \{0\} \to \mathcal{PR}.$$

We define the weighted s.c.c's by

$$\mathcal{WS} := \{t\alpha \mid t \in \mathbb{R}_{\geq 0} \text{ and } \alpha \in \mathcal{S}\}.$$

Consider the embedding

$$\mathcal{WS} \ni t\alpha \overset{i_*}{\hookrightarrow} i_*(t\alpha) := [\beta \mapsto t \cdot i(\beta, \alpha)] \in \mathcal{R}.$$

By taking the closure, we get the space $\mathcal{MF} = \mathcal{MF}(X)$ of measured foliations on X. i.e.

$$\mathcal{MF} = \overline{i_*(\mathcal{WS})} \subset \mathcal{R}.$$

We define the space $\mathcal{PMF}$ of projective measured foliations on X by

$$\mathcal{PMF} = \mathrm{proj}(\mathcal{MF} - \{0\}) \subset \mathcal{PR}.$$

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- $\mathcal{MF}$ is homeomorphic to $\mathbb{R}^{6g-6+2n}$.
- The intersection number function

$$\mathcal{WS} \times \mathcal{WS} \ni (t\alpha, s\beta) \mapsto i(t\alpha, s\beta) := ts \cdot i(\alpha, \beta)$$

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- $\mathcal{PMF}$ is homeomorphic to $S^{6g-7+2n}$.

For $\alpha \in \mathcal{S}$ and $y = (Y, f) \in T(X)$, we denote by

$$\ell_y(\alpha)$$

the hyperbolic length of the geodesic homotopic to $f(\alpha)$ on Y.
We consider the following maps

$$T(X) \ni y \overset{\tilde{\Phi}_{Th}}{\mapsto} \tilde{\Phi}_{Th}(y) := [\alpha \mapsto \ell_y(\alpha)] \in \mathcal{R} - \{0\} \overset{\text{proj}}{\longrightarrow} \mathcal{PR}.$$

Then, it is known that the composite map

$$\Phi_{Th} := \text{proj} \circ \tilde{\Phi}_{Th} : T(X) \to \mathcal{PR}$$

is embedding and its image is relatively compact in $\mathcal{PR}$.

We say that

$$\partial_{Th}T(X) := \overline{\Phi_{Th}(T(X))} - \Phi_{Th}(T(X)) \subset \mathcal{PR}.$$

is the Thurston boundary of $T(X)$.

Theorem (Thurston)

The Thurston boundary coincides with the space of projective measured foliations:

$$\partial_{Th}T(X) = \mathcal{PMF} \cong S^{6g-7+2n}.$$

in $\mathcal{PR}$. Furthremore, the Thurston compactification $\overline{\Phi_{Th}(T(X))}$ is homeomorphic to the closed ball of dimension $6g - 6 + 2n$.

Let $\alpha \in \mathcal{S}$ and $y = (Y, f) \in T(X)$. The extremal length of α on y is, by definition

$$\mathrm{Ext}_y(\alpha) = \sup_{\rho} \frac{\ell_\rho(\alpha)^2}{\int_Y \rho(z)^2 dxdy}$$

where ρ runs over all conformal measurable metrics on Y and $\ell_\rho(\alpha)$ is the ρ-length of $f(\alpha)$:

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S. Kerckhoff has shown that when we set

$$\mathrm{Ext}_y(t\alpha) = t^2 \mathrm{Ext}_y(\alpha),$$

the extremal length extends continuously on $\mathcal{MF}$.

Theorem (Hubbard-Masur)

For all $F \in \mathcal{MF}$ and $y \in T(X)$, there is a unique $q_{F,y} \in \mathcal{Q}_y$ s.t.

$$i(\alpha, F) = \inf_{\alpha' \in f(\alpha)} \int_{\alpha'} \left| \mathrm{Re}\sqrt{q} \right|$$

for all $\alpha \in \mathcal{S}$.

Then, it holds

$$\mathrm{Ext}_y(F) = \|q_{F,y}\| = \int_Y |q_{F,y}(z)| dxdy$$

for all $F \in \mathcal{MF}$ and $y \in T(X)$.

We consider

$$T(X) \ni y \mapsto \tilde{\Phi}_{GM}(y) := [\alpha \mapsto \mathrm{Ext}_y(\alpha)^{1/2}] \in \mathcal{R} - \{0\} \xrightarrow{\mathrm{proj}} \mathcal{PR}.$$

Then, F.Gardiner and H.Masur observed that the composite map

$$\Phi_{GM} := \mathrm{proj} \circ \tilde{\Phi}_{GM} : T(X) \to \mathcal{PR}$$

is embedding and its image is relatively compact in $\mathcal{PR}$.

The complement

$$\partial_{GM} T(X) = \overline{\Phi_{GM}(T(X))} - \Phi_{GM}(T(X)) \subset \mathcal{PR}$$

is called the Gardiner-Masur boundary.

F. Gardiner and H. Masur have shown that

$$\partial_{Th} T(X) \subset \partial_{GM} T(X)$$

as subsets of $\mathcal{PR}$.

Denote by $x_0 = (X, id)$ the base point.

For $y \in T(X)$, we set

$$\mathcal{E}_y(F) := \left\{ \frac{\mathrm{Ext}_y(F)}{K_y} \right\}^{1/2} : \mathcal{MF} \to \mathbb{R}_+,$$

where $K_y = \exp(2d_T(x_0, y))$, and d_T is the Teichmüller distance on $T(X)$:

$$d_T(y_1, y_2) = \frac{1}{2} \log \inf \left\{ K(h) \mid h : Y_1 \to Y_2 \text{ q.c. } h \sim f_2 \circ f_1^{-1}. \right\}$$

where $y_i = (Y_i, f_i)$ $(i = 1, 2)$.

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Thus, any boundary point of the Thurston boundary $\partial_{Th} T(X)$ is represented by a continuous function on $\mathcal{MF}$.

Theorem 1

For any $p \in \partial_{GM}T(X)$, there is a continuous function $\mathcal{E}_p$ on $\mathcal{MF}$ with the following two properties.

(1) For $t > 0$ and $F \in \mathcal{MF}$,

$$\mathcal{E}_p(tF) = t\mathcal{E}_p(F).$$

(2) The function

$$\mathcal{S} \ni \alpha \mapsto \mathcal{E}_p(\alpha)$$

represents p.

Furthermore, when $\{y_n\}_n$ converges to $p \in \partial_{GM}T(X)$, there is a subsequence $\{y_{n_j}\}_j$ and $t_0 > 0$ such that $\mathcal{E}_{y_{n_j}}$ converges to $t_0 \cdot \mathcal{E}_p$ on any compact sets of $\mathcal{MF}$.

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We ONLY need the following property of uniquely ergodic measured foliations later.

Intersection numbers and UE (Masur)

Let G be a uniquely ergodic measured foliation. If $F \in \mathcal{MF}$ satisfies

$$i(F, G) = 0$$

then $F = tG$ for some $t \geq 0$.

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Notice that for any projective class

$$[G] \in \mathcal{PMF} \cong \partial_{Th}T(X) \subset \partial_{GM}T(X)$$

is represented by the function

$$\mathcal{MF} \ni F \mapsto i(F, G) = \mathcal{E}_{[G]}(F)$$

in $\mathcal{R} = \mathbb{R}^{\mathcal{S}}_{\geq 0}$.

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in $\mathcal{R} = \mathbb{R}_{\geq 0}^{\mathcal{S}}$.

This means that uniquely ergodic projective class $[G] \in \partial_{Th}T(X)$ is a uniquely ergodic boundary point in $\partial_{GM}T(X)$.

We have the converse.

Theorem 2

For any uniquely ergodic $p \in \partial_{GM}T(X)$, there is a uniquely ergodic $G \in \mathcal{MF}$ such that

$$\mathcal{E}_p(F) = i(F, G)$$

for $F \in \mathcal{MF}$. Furthermore, G is unique up to multiplying positive constants.

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Theorem 2

For any uniquely ergodic $p \in \partial_{GM} T(X)$, there is a uniquely ergodic $G \in \mathcal{MF}$ such that

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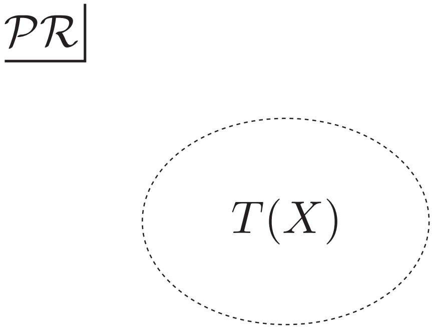
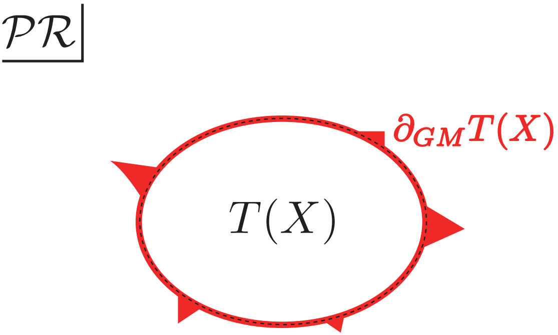
This means that any uniquely ergodic boundary point in $\partial_{GM} T(X)$ is contained in the Thurston boundary.

This is a schematic picture.



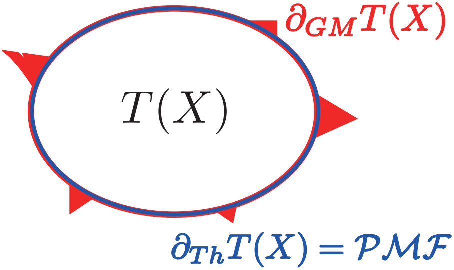
This is a schematic picture. $\partial_{GM}(X)$



N.B. I DON'T know about any topological structure of $\partial_{GM}T(X)$.

This is a schematic picture. $\partial_{Th}T(X) \subset \partial_{GM}(X)$ (Gardiner-Masur) .

$\mathcal{PR}$



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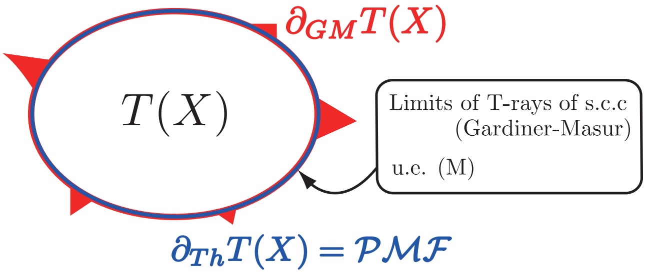

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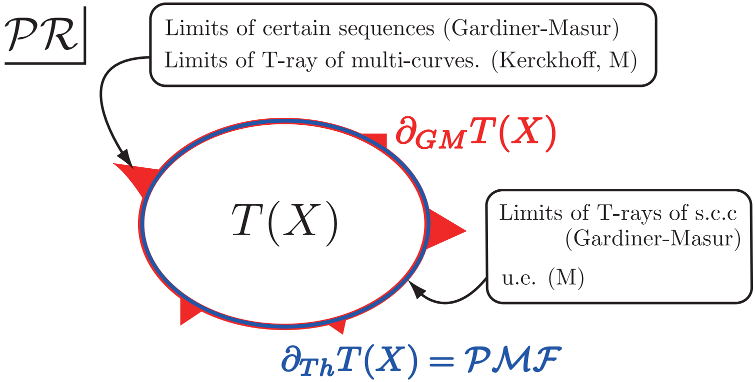

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Let S denote a surface of genus g with n punctures.
The quotient space

$$\mathtt{Flat}(S) = Q^1/q \sim e^{i\theta}q$$

is canonically identified with the space of singular flat structure (whose cone angles form $n\pi$ ($n \in \mathbb{N}$).

Let $C(S)$ be the space of geodesic currents on S, and set

$$\mathcal{P}C(S) = (C(S) - \{0\})/\mathbb{R}_+.$$

It is known that $\mathcal{MF}$ is canonically contained in $C(S)$ and the intersection number function $i(\cdot,\cdot)$ on $\mathcal{MF}$ extends continuously on $C(S)$ (Bonahon).

M. Duchin, C. Leininger, K. Rafi construct an embedding

$$\mathtt{Flat}(S) \ni q \mapsto L_q \in \mathcal{C}(S)$$

such that the q-length of $\alpha \in \mathcal{S}$ satsifies

$$\ell_q(\alpha) = i(L_q, F).$$

Furthremore, they observe that the q-length of $F \in \mathcal{MF}$ is well-defined and

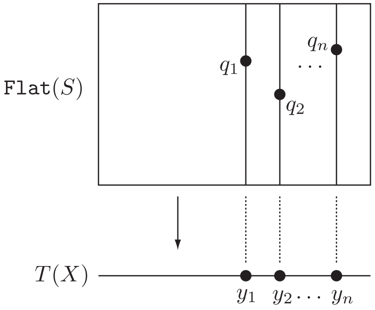
$$\mathtt{Flat}(S) \times \mathcal{MF} \ni (q, F) \mapsto \ell_q(F) = i(L_q, F).$$

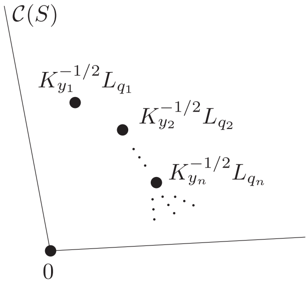
is continuous.

A sequence $\{q_n\}_{n=1}^{\infty}$ in $\mathtt{Flat}(S)$ is said to be stable if any accumulation point in $\mathcal{C}(S)$ of the sequence

$$\left\{ \frac{1}{K_{y_n}^{1/2}} L_{q_n} \right\}_{n=1}^{\infty}$$

is NOT the zero-geodesic current where $y_n \in T(X)$ is taken to satisfy $q_n \in Q_{y_n}$.





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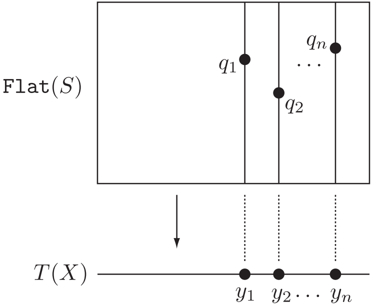

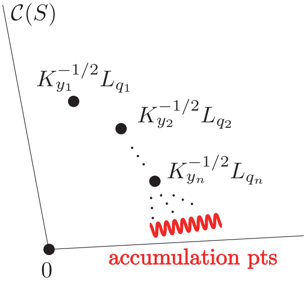

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- (Stability criterion) Let $q_n = q_{F_n, y_n}/\|q_{F_n, y_n}\|$. Suppose that $F_n \to F$, $y_n \to p$ and $\mathcal{E}_p(F) \neq 0$. Then, $\{q_n\}_{n=1}^{\infty}$ is stable.
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$$
\begin{aligned}
i(L_\infty, H) &\leq t_0\,\mathcal{E}_p(H) \quad (\forall H \in \mathcal{MF}) \\
i(L_\infty, F) &= t_0\,\mathcal{E}_p(F)
\end{aligned}
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Let p be a uniquely ergodic boundary point. Let $\{y_n\}_n \subset T(X)$ with $y_n \to p$. We may assume that $\mathcal{E}_{y_n}$ converges to $t_0\,\mathcal{E}_p$.
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Let $F \in \mathcal{MF}$ with $\mathcal{E}_p(F) \neq 0$. From (Stablility criterion), $\{q_n\}_n$ is stable, where $q_n = q_{F,y_n}/\|q_{F,y_n}\|$. We may assume that

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Then, by (Limit of stab. seq.), $L_\infty \in \mathcal{MF} - \{0\}$ and

$$i(L_\infty, H) \leq t_0 \mathcal{E}_p(H) \quad (\forall H \in \mathcal{MF})$$
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$$\begin{aligned} i(L_\infty, H) &\leq t_0\,\mathcal{E}_p(H) \quad (\forall H \in \mathcal{MF}) \\ i(L_\infty, F) &= t_0\,\mathcal{E}_p(F). \end{aligned}$$

In particular, $i(L_\infty, G) \leq \mathcal{E}_p(G) = 0$. Hence, $L_\infty = t\,G$ for some $t > 0$, and $\mathcal{E}_p(H) \neq 0$ if $H \notin \mathbb{R}\cdot G$.

Let $F' \in \mathcal{MF} - \{0\}$ with $F' \notin \mathbb{R} \cdot G$. By the previous argument, $\{q'_n\}_n$ ($q'_n = q_{F',y_n}/\|q_{F',y_n}\|$) contains a subsequence $\{q'_{n_j}\}_j$ such that

$$K_{y_{n_j}}^{-1/2} L_{q'_{n_j}} \to L'_\infty = t'\, G \in \mathcal{MF} - \{0\}.$$

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for some $t' > 0$.

Furthermore,

$$i(L'_\infty, H) \le t_0 \, \mathcal{E}_p(H) \quad (\forall H \in \mathcal{MF})$$
$$i(L'_\infty, F') = t_0 \, \mathcal{E}_p(F').$$

Now we have $L_\infty = t\,G$ and $L'_\infty = t'\,G$. Furthermore, they satisfy

$$i(L_\infty, H) \le t_0\,\mathcal{E}_p(H) \quad (\forall H \in \mathcal{MF}) \tag{1}$$
$$i(L_\infty, F) = t_0\,\mathcal{E}_p(F) \tag{2}$$
$$i(L'_\infty, H) \le t_0\,\mathcal{E}_p(H) \quad (\forall H \in \mathcal{MF}) \tag{3}$$
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From (2) and (3),

$$t'\,i(F,G) = i(L'_\infty, F) \le t_0\mathcal{E}_p(F) = t\,i(F,G).$$

Hence $t' \le t$.

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$$i(L'_\infty, H) \leq t_0\,\mathcal{E}_p(H) \quad (\forall H \in \mathcal{MF}) \tag{3}$$

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$$t'\,i(F,G) = i(L'_\infty, F) \leq t_0\mathcal{E}_p(F) = t\,i(F,G).$$

Hence $t' \leq t$. From (1) and (4),

$$t\,i(F',G) = i(L_\infty, F') \leq t_0\mathcal{E}_p(F') = t'\,i(F',G).$$

Hence $t' = t$.

Now we have $L_\infty = t\,G$ and $L'_\infty = t'\,G$. Furthermore, they satisfy

$$i(L_\infty, H) \le t_0\,\mathcal{E}_p(H) \quad (\forall H \in \mathcal{MF}) \tag{1}$$

$$i(L_\infty, F) = t_0\,\mathcal{E}_p(F) \tag{2}$$

$$i(L'_\infty, H) \le t_0\,\mathcal{E}_p(H) \quad (\forall H \in \mathcal{MF}) \tag{3}$$

$$i(L'_\infty, F') = t_0\,\mathcal{E}_p(F'). \tag{4}$$

From (2) and (3),

$$t'\,i(F,G) = i(L'_\infty, F) \le t_0\mathcal{E}_p(F) = t\,i(F,G).$$

Hence $t' \le t$. From (1) and (4),

$$t\,i(F',G) = i(L_\infty, F') \le t_0\mathcal{E}_p(F') = t'\,i(F',G).$$

Hence $t' = t$. This means that t and t' are independent of F and F' and hence

$$t_0\mathcal{E}_p(F) = t\,i(F,G)$$

for all $F \in \mathcal{MF}$.

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- Thank you very much for your attention.