Generalized Teichmüller Spaces and Moduli of Geometric Structures

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July 19, 2010

Teichmüller Space

$$\begin{aligned}
\mathcal{T}(\Sigma_g) &= \{(X,f) \mid X \text{ Riemannian of curvature } -1,\ f:\Sigma_g \overset{\sim}{\to} X\}/\sim \\
&= \{\rho : \Gamma_g = \pi_1(\Sigma_g) \to \mathrm{SO}(2,1) \mid \rho \text{ is discrete and faithful}\}/\sim \\
&= \{\rho : \Gamma_g = \pi_1(\Sigma_g) \to \mathrm{SO}(2,1) \mid \exists \xi : \partial_\infty \widetilde{\Sigma}_g \to \partial_\infty \mathbb{H}^2, \\
&\qquad\qquad\qquad\qquad \text{continuous, injective and equivariant}\} \\
&= e^{-1}(2g-2)
\end{aligned}$$

where e is the Euler number (Goldman 1988).

Generalization I: Hitchin's component

Let G be a semi-simple split real Lie group (i.e. $G = \mathrm{SL}(n, \mathbf{R})$). The Hitchin's component $\mathcal{H}(\Gamma_g, G)$ is the connected component of $\mathcal{X}(\Gamma_g, G)$ (the character variety) that contains (the class of) the representation $\tau \circ \iota$ where

- $\iota : \Gamma_g \to \mathrm{SL}(2, \mathbf{R})$ is discrete and faithful,
- and $\tau : \mathrm{SL}(2, \mathbf{R}) \to G$ is the principal $\mathrm{SL}(2, \mathbf{R})$ (i.e. $\tau : \mathrm{SL}(2, \mathbf{R}) \to \mathrm{SL}(n, \mathbf{R})$ is the irreducible n-dimensional representation).

Theorem (Hitchin 1992)

$$\mathcal{H}(\Gamma_g, G) \simeq \mathbf{R}^{(2g-2)\dim G}.$$

Generalization II: Maximal Representations

G is a semi-simple real Lie group of Hermitian type (i.e. $G = \mathrm{Sp}(2n, \mathbf{R})$).
$\mathcal{M}(\Gamma_g, G)$ is the space of (conjugacy class of) maximal representations $\Gamma_g \to G$.

Question: Are $\mathcal{H}$ and $\mathcal{M}$ moduli space of (G, X)-structures (maybe with additional properties) ?

Example: (Goldman 90, Goldman-Choi 93) $\mathcal{H}(\Gamma_g, \mathrm{SL}(3, \mathbf{R}))$ is the moduli space of marked convex projective structures on Σ_g.

Today: We shall construct embeddings into moduli spaces of geometric structures and shows that the images are union of connected components.

Key input

Theorem

The spaces $\mathcal{H}(\Gamma_g, G)$ and $\mathcal{M}(\Gamma_g, G)$ are contained is the space of Anosov representations.

due to: Labourie for $\mathcal{H}(\Gamma_g, \mathrm{SL}(n, \mathbf{R}))$, Fock and Goncharov for $\mathcal{H}(\Gamma_g, G)$, Burger, Iozzi, Labourie and Wienhard for $\mathcal{M}(\Gamma_g, \mathrm{Sp}(2n, \mathbf{R}))$, Burger, Iozzi and Wienhard for $\mathcal{M}(\Gamma_g, G)$.

Remarks:

- the inclusion is strict unless $G = \mathrm{SL}(2, \mathbf{R})$.
- The terminology is due to F. Labourie.

If $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbf{R})$ is Anosov, put
$\Omega_\rho = \mathbb{S}^{2n-1} \smallsetminus \bigcup_{t \in \partial_\infty \tilde{\Sigma}_g} \xi(t) = \mathbb{S}^{2n-1} \smallsetminus K$ with $K \simeq \mathbb{S}^1 \times \mathbb{S}^{n-1}$.

Theorem (G., Wienhard)

The action of $\rho(\Gamma_g)$ on Ω_g is free, proper and co-compact.
The topology of $\rho(\Gamma_g)\backslash\Omega_\rho$ is (locally) constant: If $N \subset \{Anosov\}$ is connected, then $\bigcup_{\rho \in N} \rho(\Gamma_g)\backslash\Omega_\rho \simeq N \times \rho_0(\Gamma_g)\backslash\Omega_{\rho_0}$.

Proposition

If ρ is in $\mathcal{M}(\Gamma_g, \mathrm{Sp}(2n, \mathbf{R}))$, then $\rho(\Gamma_g)\backslash\Omega_\rho$ is a $\mathrm{SO}(n)/\mathrm{SO}(n-2)$-bundle over Σ_g.

Let $X(M)$ be the moduli space of marked (G, X)-structures on M.

Corollary

Let $\mathcal{C}$ be a connected component of $\mathcal{M}(\Gamma_g, \mathrm{Sp}(2n, \mathbf{R}))$, then the association $\rho \mapsto \rho(\Gamma_g)\backslash\Omega_\rho$ induces an embedding $\mathcal{C} \to \mathbb{S}^{2n-1}(M)$ whose image is a connected component.