Moduli spaces of hyperbolic surfaces with cone angles.

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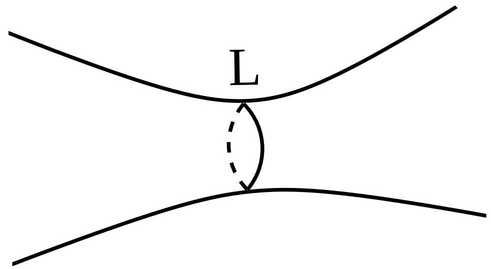
Summary.

- Define moduli spaces of hyperbolic surfaces with cone angles.
- These are equipped with symplectic forms and hence have well-defined volumes depending on the cone angles.
- Mirzakhani proved that volumes of moduli spaces of hyperbolic surfaces with geodesic boundary lengths are polynomial in the lengths.
- The volume polynomial analytically continues to give volumes of moduli spaces of hyperbolic surfaces with small cone angles.
- **Question:** how are Mirzakhani's volume polynomials related to the volumes of moduli spaces of hyperbolic surfaces with large cone angles?

$\mathcal{M}_{g,n}(L_1, ..., L_n)$ = moduli space of oriented hyperbolic surfaces with length L_i geodesic boundary components.

Model of a neighbourhood of a length L closed geodesic



$$= \frac{\mathbb{H}^2}{\langle g_L \rangle}$$

$$g_L(L^2) = \begin{pmatrix} \cosh\frac{L}{2} & \frac{2}{L}\sinh\frac{L}{2} \\ \frac{L}{2}\sinh\frac{L}{2} & \cosh\frac{L}{2} \end{pmatrix}, \text{ fixed points } \frac{\pm 2}{\sqrt{L^2}} \in \overline{\mathbb{H}^2}$$

Generalise $L^2 > 0$ to $L^2 \in \mathbb{R}$.

$L^2 > 0$—closed geodesic, $\quad L^2 = 0$—cusp, $g_L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

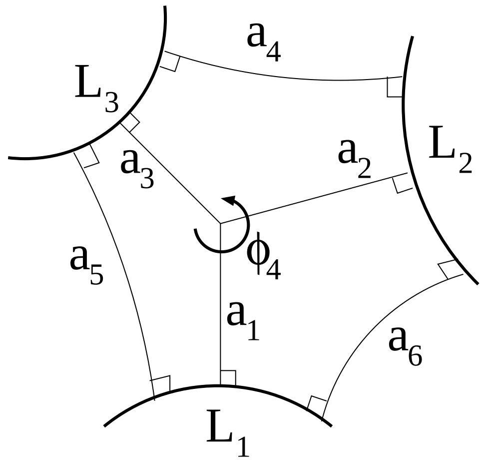
$L^2 < 0$—cone angle, g_L rotation by ϕ for $L = i\phi$

$\mathcal{M}_{g,n}(L_1, ..., L_n)$= moduli space of oriented hyperbolic surfaces with geodesic boundary components, cusps and cone angles corresponding to $L_j = i\phi_j$.

Different behaviours

- all $L_j = 0$ (cusps)
- all $L_j = i\phi_j$, $0 \leq \phi_j < 2\pi$
- $L_j > 0$ or $L_j = i\phi_j$, $\phi_j < \pi$

Arc lengths $\{a_i\}$ give (generalised) Penner coordinates.



Poisson structure on $\mathcal{M}_{g,n}(\,\cdot\,,\ldots,\,\cdot\,)$.

$$\eta_{WP} = \sum_{j=1}^{n} \sum_{k,l} \frac{\sinh(\alpha_{j,kl} L_j/2)}{\sinh(L_j/2)} \frac{\partial}{\partial a_k} \wedge \frac{\partial}{\partial a_l} \qquad \text{(Mondello)}$$

$$\alpha_{j,kl} = 1 - 2\times(\text{fraction of rotation around } L_j \text{ between arcs})$$

η_{WP} is degenerate—non-degenerate on $L_j = \text{constant}$

ω_{WP} dual Weil-Petersson symplectic form

$$V_{g,n}(L_1, \ldots, L_n) = \int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \frac{\omega_{WP}^{3g-3+n}}{(3g-3+n)!}$$

Theorem (Mirzakhani) *$V_{g,n}(L_1, ..., L_n)$ is polynomial in L_i^2.*

Uses a McShane identity.

True for $L_j \geq 0$, $L_j = i\phi_j$, $\phi_j \leq \pi$. (Tan-Wong-Zhang)

Q. How is $V_{g,n}(L_1, ..., L_n)$ related to the volume of the moduli space for cone angles $> \pi$?

Example. $V_{0,4}(L_1, ..., L_4) = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2 + 4\pi^2)$

does not give the volume for large enough angles.

Guess: the polynomial gives the volume when there is only

one cone angle ($< 2\pi$.)

Theorem (Norman Do, N.)

$$(1) \qquad V_{g,n+1}(L_1, ..., L_n, 2\pi i) = \sum_{k=1}^{n} \int_0^{L_k} L_k V_{g,n}(L_1, ..., L_n) dL_k$$

$$(2) \qquad \frac{\partial V_{g,n+1}}{\partial L_{n+1}}(L_1, ..., L_n, 2\pi i) = 2\pi i(2g - 2 + n)V_{g,n}(L_1, ..., L_n)$$

For $0 \leq \phi_j < 2\pi$ there exists a forgetful map

$$\mathcal{M}_{g,n+1}(i\phi_1, ..., i\phi_n, i\phi_{n+1}) \to \mathcal{M}_{g,n}(i\phi_1, ..., i\phi_n).$$

As $\phi_{n+1} \to 2\pi$ the Kähler metric degenerates along fibres and tends to the pull-back of the Kähler metric downstairs. (Schumacher-Trapani)

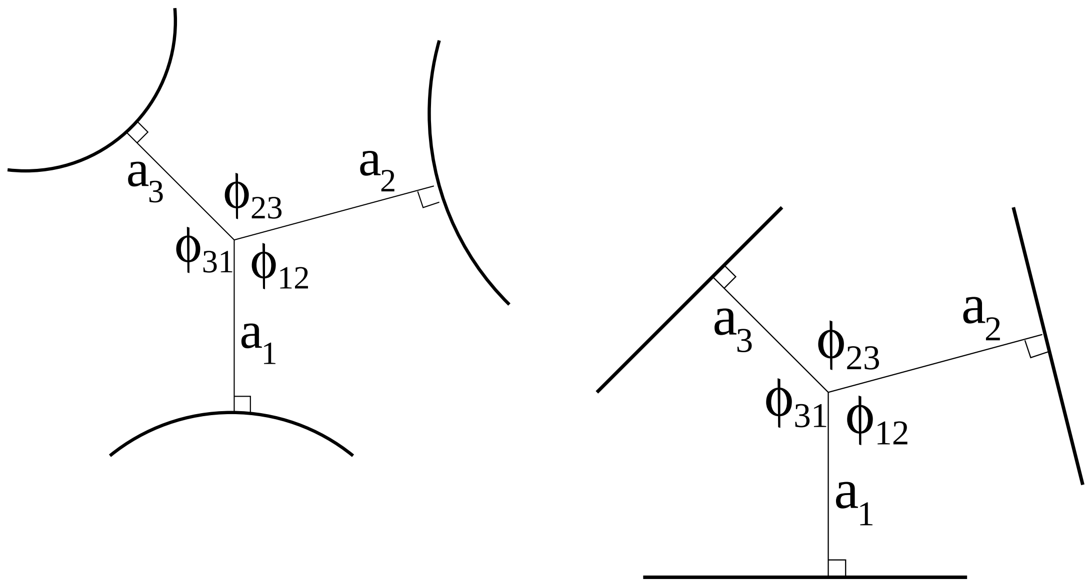
Specialise (1) to

$$(3) \qquad V_{g,n+1}(0, ..., 0, 2\pi i) = 0.$$

Study the degeneration as $\phi_{n+1} \longrightarrow 2\pi$.

$$\sin\frac{\phi_{n+1}}{2}\cdot\eta_{WP}\longrightarrow\sum_{k,l}\sin(\phi_{n+1,kl})\frac{\partial}{\partial a_k}\wedge\frac{\partial}{\partial a_l}.$$

Elementary geometry.



$\{a_i, a_j\} = \sin\phi_{ij}$

Lengths a_i are functions on the hyperbolic surface. Hyperbolic metric (Kähler) gives Poisson structure η_{hyp}.

The uniform convergence

$$\sin\frac{\phi}{2}\cdot\eta_{WP}\longrightarrow\eta_{hyp}$$

almost gives (3) and (2).

Idea

$$\omega_{WP,g,n+1}\sim\omega_{WP,g,n}+\sin\frac{\phi}{2}\cdot\omega_{hyp}$$

For $N=3g-3+n$,

$$\frac{\omega_{WP,g,n+1}^{N+1}}{(N+1)!}\sim(N+1)\frac{\omega_{WP,g,n}^{N}}{(N+1)!}\cdot\sin\frac{\phi}{2}\cdot\omega_{hyp}$$

which should integrate to give

$$\mathrm{Vol}_{g,n+1}\sim 4\pi(2g-2+n)\sin\frac{\phi}{2}\cdot\mathrm{Vol}_{g,n}(L_1,...,L_n).$$

Eynard and Orantin also (rigorously) prove (1) and (2).

- A model / B model mirror picture
- A model side: $V_{g,n}(L_1, ..., L_n)$—generating function for Gromov-Witten invariants with Kähler parameters as variables.
- B model side: Laplace transform of $V_{g,n}(L_1, ..., L_n)$
  - Underlying the B model is a Riemann surface Σ equipped with a meromorphic 1-form θ and a map $\Sigma \to S^2$.
  - B model is concerned with variations of periods of θ.
  - Equations (1) and (2) are special cases of general properties of the B model.