Margulis Numbers of Hyperbolic 3-Manifolds

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Margulis numbers

M a hyperbolic n-manifold

Write $M = \mathbb{H}^n/\Gamma$

$\Gamma \leq \mathrm{Isom}(\mathbb{H}^n)$ discrete, torsion-free, uniquely determined up to conjugacy by the hyerbolic structure of M

I'll always assume Γ is non-elementary (i.e. has no abelian subgroup of finite index)

Definition

A *Margulis number* for M (or for Γ) is a $\mu > 0$ such that: If $P \in \mathbb{H}^n$, the elements $x \in \Gamma$ such that $d(P, x \cdot P) < \mu$ generate an elementary subgroup.

Here d denotes hyperbolic distance on $\mathbb{H}^n$.

Margulis numbers, cont'd

If M is closed, or 2-dimensional, or 3-dimensional and orientable, all elementary subgroups of Γ are abelian. Thus the condition in the definition of a Margulis number becomes: If $P \in \mathbb{H}^n, \quad x, y \in \Gamma$, and $\max(d(P, x \cdot P), d(P, y \cdot P)) < \mu$, then x and y commute.

Margulis constants

The *Margulis Lemma* implies that for every $n \geq 2$ there is a positive constant which is a Margulis number for every hyperbolic n-manifold. The largest such number, $\mu(n)$, is called the *Margulis constant* for hyperbolic n-manifolds.

It is known that

- $\mu(3) \geq 0.104\ldots$ (Meyerhoff)
- $\mu(3) \leq 0.65\ldots$ (Culler)

For every $n \geq 2$, Kellerhals has shown that

$$\mu(n) \geq \frac{2^{\nu+1}}{3^{\nu+1}\pi^{\nu}} \frac{\Gamma(\frac{\nu+2}{2})^2}{\Gamma(\nu+2)},$$

where $\nu = [\frac{n-1}{2}]$.

Margulis numbers and geometry

Suppose M^3 is hyperbolic and (for simplicity) closed and orientable.

A Margulis number μ for M determines a canonical decomposition of M into a *μ-thin part*, consisting of tubes around closed geodesics, and a *μ-thick part*, a 3-manifold with torus boundary components, consisting of points where the injectivity radius is at least $\mu/2$.

There are only finitely many topological possibilities for the μ-thick part of M given an upper bound on the volume of M. Topologically, M is obtained by a Dehn filling from its μ-thick part.

This makes estimation of the maximal Margulis number for M a crucial step in understanding the geometric structure of M. The larger μ is, the fewer possibilities there are for the μ-thick part.

A topological theorem

Theorem (Jaco-S.)

Let M be a hyperbolic 3-manifold (possibly with cusps and possibly of infinite volume). Let $J \leq \pi_1(M)$ be a subgroup of rank at most two which has infinite index in $\pi_1(M)$. Then J is either an abelian group or a free group of rank 2.

This is a topology theorem. The proof uses the compact core theorem and a characterization of free groups due to Magnus.

The $\log(2k-1)$ Theorem

Theorem (Anderson-Canary-Culler-S. + Marden Conjecture (Agol and Calegari-Gabai) + Bers Density Conjecture (Bromberg et al.))

Let $k \geq 2$ be an integer and let F be a discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$ which is freely generated by elements $\xi_1, \ldots, \xi_k$. Let P be any point of $\mathbb{H}^3$ and set $d_i = \mathrm{dist}(P, \xi_i \cdot P)$ for $i = 1, \ldots, k$. Then we have

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} \leq \frac{1}{2}.$$

In particular there is some $i \in \{1, \ldots, k\}$ such that $d_i \geq \log(2k-1)$.

(Note the curious similarity to McShane's identity.)

The $\log(2k-1)$ theorem, cont'd

The proof of the $\log(2k-1)$ theorem involves the construction of the Patterson-Sullivan measure, the Banach-Tarski decomposition of a free group, and deep results from the theory of Kleinian groups.

The $\log 3$ theorem and the result of J-S stated immediately imply:

- $\log 3 = 1.09\ldots$ is a Margulis number for any closed, orientable hyperbolic 3-manifold M such that every subgroup of rank at most 2 in $\pi_1(M)$ has infinite index. In particular this holds if $H_1(M;\mathbb{Q})$ has rank at least 3, or if $H_1(M;\mathbb{Z}_p)$ has rank at least 4 for some prime p (S.-Wagreich).

Haken manifolds

A compact, orientable, irreducible (e.g. hyperbolic) 3-manifold M is called a *Haken manifold* if it contains a properly embedded orientable surface S which is *incompressible* in the sense that (i) S is not a 2-sphere and (ii) the inclusion homomorphism $\pi_1(S) \to \pi_1(M)$ is injective.

Theorem (Culler-S.)

Let M be a hyperbolic 3-manifold which is homeomorphic to the interior of a Haken manifold. (In particular M may be a closed Haken manifold.) Then 0.286 is a Margulis number for M. If $H_1(M; \mathbb{Q}) \neq 0$, then 0.292 is a Margulis number for M.

One novel feature of the proof of this result is that it involves a decomposition of the Patterson-Sullivan measure for groups that are not necessarily free.

Generic Margulis numbers

Theorem (S.)

Up to isometry there are at most finitely many closed, orientable hyperbolic 3-manifolds for which 0.292 is not a Margulis number.

This may be expressed as saying that 0.292 is a "generic Margulis number" for closed hyperbolic 3-manifolds.

This theorem is deduced from the above result about Haken manifolds by the use of the representation variety of a two-generator free group. I will being giving a similar argument in detail a little later.

Margulis Numbers and Volume Bounds, I

Theorem A (S.)

Let λ be a positive real number strictly less than $\log 3$. Then there is a constant V_λ such that every closed, orientable hyperbolic 3-manifold of volume greater than V_λ admits λ as a Margulis number.

Corollary

Let λ be a positive real number strictly less than $\log 3$. Then there is a there is a natural number d_λ such that for every closed, orientable hyperbolic 3-manifold M, either $\pi_1(M)$ has a rank-2 subgroup of index at most d_λ, or M admits λ as a Margulis number.

Margulis Numbers and Volume Bounds, I, cont'd

Corollary

Let λ be a positive real number strictly less than $\log 3$. Then there is a there is a natural number k_λ such that every closed, orientable hyperbolic 3-manifold whose fundamental group has rank greater than k_λ admits λ as a Margulis number.

Margulis Numbers and Volume Bounds, II

Given λ with $0 < \lambda < (\log 3)/2$

For a large enough integer $N > 0$ we have

$$\frac{3^{N+1} - 1}{4N + 1} \geq 2667(\sinh(2N\lambda + .104) - (2N\lambda + .104)).$$

Let $N(\lambda)$ denote the smallest such positive integer N.

As $\lambda \to (\log 3)/2$ from below, $N(\lambda)$ grows a little faster than $1/((\log 3) - 2\lambda)$.

Theorem B (S.)

Let λ be a positive real number strictly less than $(\log 3)/2$. Then every closed, orientable hyperbolic 3-manifold M with

$$\operatorname{vol} M > \lambda \cdot (8N(\lambda) - 2)$$

admits λ as a Margulis number.

Margulis Numbers and Volume Bounds, II, cont'd

Let $V_0 = 0.94\ldots$ denote the volume of the Weeks manifold.

Corollary

Let λ be a positive real number strictly less than $(\log 3)/2$. Then for every closed, orientable hyperbolic 3-manifold M, either $\pi_1(M)$ has a rank-2 subgroup of index at most $\lambda \cdot (8N(\lambda) - 2)/V_0$, or M admits λ as a Margulis number.

Corollary

Let λ be a positive real number strictly less than $(\log 3)/2$. Then every closed, orientable hyperbolic 3-manifold M with

$$\operatorname{rank} \pi_1(M) > 2 + \log_2(\lambda \cdot (8N(\lambda) - 2)/V_0)$$

admits λ as a Margulis number.

Margulis Numbers and Trace Fields

Theorem (S.)

Let K be any number field, and let D denote its degree. The number of (isometry classes of) closed, non-arithmetic hyperbolic 3-manifolds which are $\mathbb{Z}_6$-homology 3-spheres, have trace field K, and do not admit 0.183 as a Margulis number is at most $141 \times 2^{24(D+1)}$.

(Recall that by result about "generic Margulis numbers" that I mentioned earlier, there are at most finitely many closed, orientable hyperbolic 3-manifolds (up to isometry) for which 0.292 is not a Margulis number.)

The proof of the theorem stated above depends on the $\log 3$ theorem, the algebra of congruence subgroups, Beukers and Schlickewei's explicit form of Siegel and Mahler's finiteness theorem for solutions to the unit equation in number fields, and Theorem B.

Proof of Theorem A

Recall the statement:

Theorem A (S.)

Let λ be a positive real number strictly less than $\log 3$. Then there is a constant V_λ such that every closed, orientable hyperbolic 3-manifold of volume greater than V_λ admits λ as a Margulis number.

We reason by contradiction. Assume there is a sequence $(M_i)_{i\geq 1}$ of closed, orientable hyperbolic 3-manifolds such that $\operatorname{vol} M_i \to \infty$ and no M_i admits λ as a Margulis number.

For each i write $M_i = \mathbb{H}^3/\Gamma^{(i)}$ for some torsion-free cocompact discrete subgroup $\Gamma^{(i)}$ of $\mathrm{Isom}_+(\mathbb{H}^3)$. Then, by definition, for each i there exist non-commuting elements $x_i, y_i \in \Gamma^{(i)}$ and a point $P_i \in \mathbb{H}^3$ such that

$$\max(d(P_i, x_i \cdot P_i), d(P_i, y_i \cdot P_i)) < \lambda.$$

Proof of Theorem A, cont'd

After replacing each Γ_i by a suitable conjugate of itself in $\mathrm{Isom}_+(\mathbb{H}^3)$, we may assume that the P_i are all the same point of $\mathbb{H}^3$, which I will denote by P. Thus for each i we have

$$\max(d(P, x_i \cdot P), d(P, y_i \cdot P)) < \lambda. \qquad (1)$$

Since $\lambda < \log 3$, the $\log 3$ Theorem implies that $\widetilde{\Gamma}_i := \langle x_i, y_i \rangle$ is not free. By the result of Jaco-S. I mentioned earlier, $\widetilde{\Gamma}_i$ has finite index in Γ_i. So $\widetilde{M}_i := \mathbb{H}^3/\widetilde{\Gamma}_i$ is a closed hyperbolic 3-manifold, and $\mathrm{vol}\,\widetilde{M}_i \geq \mathrm{vol}\,M_i$. In particular, $\mathrm{vol}\,\widetilde{M}_i \to \infty$.

For each i we define a representation ρ_i of the rank-2 free group $F_2 = \langle \xi, \eta \rangle$ by $\rho_i(\xi) = x_i$, $\rho_i(\eta) = y_i$. It follows from (1) that the ρ_i lie in a compact subset of the representation variety $R = \mathrm{Hom}(F_2, \mathrm{PSL}_2(\mathbb{C}))$. Hence after passing to a subsequence we may assume the sequence (ρ_i) converges, say to ρ_∞. Set $x_\infty = \rho_\infty(\xi)$, $y_\infty = \rho_\infty(\eta)$. By (1) we have

$$\max(d(P, x_\infty \cdot P), d(P, y_\infty \cdot P)) \leq \lambda. \qquad (2)$$

Proof of Theorem A, cont'd

A theorem due to T. Jorgensen and P. Klein implies that the set D of representations of F_2 with discrete, torsion-free, non-elementary image is closed in R. Hence $\rho_\infty \in D$.

Let Φ denote the subset of D consisting of those discrete torsion-free representations whose images have finite covolume. It is well known that the function $\rho \mapsto \mathrm{vol}(\mathbb{H}^3)/\rho(F_2)$ is continuous on Φ. If $\rho_\infty \in \Phi$, it follows that

$$\mathrm{vol}(\mathbb{H}^3)/\rho_i(F_2) \to \mathrm{vol}(\mathbb{H}^3)/\rho_\infty(F_2),$$

a contradiction since

$$\mathrm{vol}(\mathbb{H}^3)/\rho_i(F_2) = \mathrm{vol}(\mathbb{H}^3)/\Gamma_i \to \infty.$$

Proof of Theorem A, concluded

If $\rho_\infty \notin \Phi$, so that $M_\infty = \mathbb{H}^3/\rho_\infty(F_2)$ has infinite volume, the proof of the Jaco-S. result shows that $\pi_1(M_\infty) \cong \rho_\infty(F_2) = \langle x_\infty, y_\infty \rangle$ is free of rank 2. The $\log 3$ Theorem then gives

$$\max(d(P, x_\infty \cdot P), d(P, y_\infty \cdot P)) \geq \log 3,$$

a contradiction to (2).

Proof of Theorem B

Recall the statement:

Theorem B (S.)

Let λ be a positive real number strictly less than $(\log 3)/2$. Then every closed, orientable hyperbolic 3-manifold M with

$$\mathrm{vol}\, M > \lambda \cdot (8N(\lambda) - 2)$$

admits λ as a Margulis number.

Proof of Theorem B, cont'd

It follows formally from two propositions:

Proposition 1

Let $M = \mathbb{H}^3/\Gamma$ *be a closed, orientable, hyperbolic 3-manifold, let* $\lambda < (\log 3)/2$ *be given, and let* x *and* y *be non-commuting elements of* Γ *such that* $\max(d(P, x \cdot P), d(P, y \cdot P)) < \lambda$ *for* $i = 1, 2$*. Then there is a reduced word* W *in two letters, with* $0 < \mathrm{length}\, W \leq 8N(\lambda)$*, such that* $W(x, y) = 1$*. Furthermore,* $\langle x, y \rangle$ *has finite index in* Γ*.*

Proposition 2

Let $(\widetilde{M}, \star)$ *be a based closed, orientable, hyperbolic 3-manifold such that* $\pi_1(\widetilde{M}, \star)$ *is generated by two elements* x *and* y*. Let* $\lambda > 0$ *be given, and suppose that* x *and* y *are represented by closed loops of length* $< \lambda$ *based at* $\star$*. Let* W *be a non-trivial reduced word in two letters such that* $W(x, y) = 1$*. Then*

$$\mathrm{vol}\, \widetilde{M} < (\mathrm{length}(W) - 2) \min(\pi, \lambda).$$

Proof of Proposition 1

Set $\mu = 0.104 < \mu(3)$. Set $N = N(\lambda)$.

The elements $\gamma \in \Gamma$ such that $d(\gamma \cdot P, P) < \mu$ generate a cyclic subgroup C of Γ (trivial or cyclic)

First look at the case C trivial. In this case I'll show there is a reduced word W in two letters, with $0 < \operatorname{length} W \leq 2N$, such that $W(x, y) = 1$.

Suppose no such W exists. Then for distinct reduced words V, V' of length at most N we have $V(x, y) \neq V'(x, y)$. Since $C = \{1\}$ it follows that $d(V(x, y) \cdot P, V'(x, y) \cdot P) \geq \mu$. So if B denotes the (open) ball of radius $\mu/2$ about P, the balls $V(x, y) \cdot B$, where V ranges over reduced words of length at most N, are pairwise disjoint.

The case where C is trivial, cont'd

There are $2(3^{N+1}-1)$ reduced words of length at most N

If V is such a word, the ball $V(x,y)\cdot B$ has volume $\beta := \mathrm{vol}\, B = 0.000589\ldots$ and is contained in the ball of radius $N\lambda + (\mu/2)$ about P, which has volume $\pi(\sinh(2N\lambda+\mu) - (2N\lambda+\mu))$. So

$$2(3^{N+1}-1)\beta < \pi(\sinh(2N\lambda+\mu) - (2N\lambda+\mu)),$$

which is a contradiction, because the definition of $N = N(\lambda)$ implies that

$$\frac{2(3^{N+1}-1)}{4N+1} > \frac{\pi}{\beta}(\sinh(2N\lambda+\mu) - (2N\lambda+\mu)).$$

The case where C is infinite cyclic

In this case I'll show there is a reduced word W in two letters, with $0 < \mathrm{length}\, W \leq 8N$, such that $W(x, y) = 1$. Suppose no such W exists.

Let V, V' be reduced words of length at most N. If $V(x, y)$ and $V'(x, y)$ represent distinct cosets of C we have

$$d(V(x, y) \cdot P, V'(x, y) \cdot P) \geq \mu,$$

so the balls $V(x, y) \cdot B$ and $V'(x, y) \cdot B$ are disjoint. I claim there are at most $4N + 1$ reduced words of length at most N representing a given coset. This will imply that

$$\frac{2(3^{N+1} - 1)\beta}{4N + 1} < \sinh(2N\lambda + \mu) - (2N\lambda + \mu)$$

which still contradicts the definition of $N = N(\lambda)$.

Proof of the Claim

We must show there are at most $4N+1$ reduced words of length at most N representing a given coset of C. If V and V' are distinct reduced words of length at most N representing the same coset, then $U = V^{-1}V'$ is equal in the rank-2 free group F_2 to a non-trivial reduced word of length at most $2N$, and $U(x,y) \in C$. So we may assume there is at least one non-trivial reduced word U_0 of length at most $2N$ such that $U_0(x,y) \in C$. Let $\hat{C}$ denote the maximal cyclic subgroup of F_2 that contains U_0.

Suppose that a given coset of C in Γ is represented by an element $V_1(x,y)$, where V_1 is a reduced word of length at most N. If V is any reduced word of length at most N, such that $V(x,y)$ belongs to the coset $V_1(x,y)C$, then $U := V^{-1}V_1$ is a word of length $2N$ and $U(x,y) \in C$. Since C is abelian, the word $W := UU_0U^{-1}U_0^{-1}$ satisfies $W(x,y) = 1$. Since W has length at most $8N$, our assumption implies that W represents the identity element of F_2. Hence $U \in \hat{C}$.

Proof of the Claim, concluded

Thus $V \mapsto V^{-1}V_1$ is an injection from the set of reduced words of length at most N representing elements of the coset $V_1(x, y)C$ to the set of elements of $\hat{C}$ represented by words of length at most $2N$. Since $\hat{C}$ is cyclic there are at most $4N + 1$ such elements.

Proof of Proposition 2

The basic method is due to Cooper.

Recall the statement (in slightly changed notation):

Proposition 2

Let $(M, \star)$ be a based closed, orientable, hyperbolic 3-manifold such that $\pi_1(M, \star)$ is generated by two elements x and y. Let $\lambda > 0$ be given, and suppose that x and y are represented by closed loops of length $< \lambda$ based at $\star$. Let W be a non-trivial reduced word in two letters such that $W(x, y) = 1$. Then

$$\mathrm{vol}\, M < (\mathrm{length}(W) - 2) \min(\pi, \lambda).$$

We have a surjection from the one-relator group $F_2/\langle\langle W \rangle\rangle$ to $\pi_1(M, \star)$ which takes the generators to x and y. We realize it by a map $f : K \to M$, where K is a complex with one vertex, two 1-cells and one 2-cell. Each 1-cell maps to a loop which is geodesic except at the base point, and has length $< \lambda$.

Proof of Proposition 2, cont'd

We subdivide the 2-cell into combinatorial triangles, each of which has all its vertices at the original 0-cell. This introduces new edges but no new vertices, and each triangle has at least one side in the original 1-skeleton. After a homotopy we may assume that each open edge of the subdivided complex maps onto a geodesic path, and that each open triangle maps onto a (possibly singular) totally geodesic triangle in M. For simplicity I'll assume f is in general position, i.e. has at most one-dimensional self-intersections.

The area of a hyperbolic triangle is less than $\min(\pi, l)$, where l is the length of the shortest side. Since K contains $\mathrm{length}(W) - 2$ combinatorial triangles, it follows that

$$\mathrm{area}\, f(K) < (\mathrm{length}(W) - 2) \min(\pi, \lambda).$$

Hence it suffices to prove that $\mathrm{vol}\, M < \mathrm{area}\, f(K)$. This will follow from two lemmas.

A topological lemma

Topological Lemma (S.)

Let M be a closed, orientable 3-manifold, let K be a connected 2-complex whose fundamental group has rank 2 but is not a rank-2 free group, and let $f : K \to M$ be a continuous map such that $f_\sharp : \pi_1(K) \to \pi_1(M)$ is surjective. Then for each component C of $M - f(K)$, the inclusion homomorphism $\pi_1(C) \to \pi_1(M)$ has cyclic image.

The proof uses the result due to Jaco-S. that I mentioned earlier, and the characteristic submanifold theory.

An isoperimetric lemma; proof of Prop. 2 concluded

Isoperimetric Lemma (Agol-Liu)

Let M be a hyperbolic 3-manifold and let C be a precompact subset of M such that the frontier F of M is piecewise smooth. Suppose that the inclusion homomorphism $\pi_1(C) \to \pi_1(M)$ has abelian image. Then

$$\mathrm{vol}\, C < \frac{1}{2}\,\mathrm{area}\, F.$$

In the notation of the proof of Proposition 2, these two lemmas imply that if C is any component of $M - f(K)$ and F_C denotes its frontier, we have $\mathrm{vol}\, C < \frac{1}{2}\,\mathrm{area}\, F_C$. Summing over the components of $M - f(K)$, we find that

$$\mathrm{vol}\, M = \sum_C \mathrm{vol}\, C < \frac{1}{2} \sum_C \mathrm{area}\, F_C = \mathrm{area}\, f(K),$$

as required.