Penner's coordinate-system for a representation space of a punctured surface group and its applications

Toshihiro Nakanishi (Shimane University)

1 Integer Solutions of Some Diophantine Equations

1.1 The Markoff equation

The Markoff equation

$$x^2 + y^2 + z^2 = 3xyz \qquad (1)$$

admits infinitely many positive integer solutions: $(x, y, z) = (1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 5)$, $(1, 5, 13)$, $(2, 5, 29)$, $(1, 13, 34)$, $(5, 13, 194)$, $(25, 29, 433)$,..., etc.

Let $\Gamma_{1,1}$ be the group generated by the rational transformations:

$$\omega_1(x, y, z) = (z, y, \frac{y^2 + z^2}{x}), \quad \omega_2(x, y, z) = (x, z, \frac{x^2 + z^2}{y}).$$

Then all positive integer solutions of (1) are in the $\Gamma_{1,1}$-orbit of $(1, 1, 1)$ (A. A. Markoff). Note that $\omega_1(x, y, z) = (z, y, 3yz - x)$,$\omega_2(x, y, z) = (x, z, 3xz - y)$ for (x, y, z) satisfying (1).

1.2 Other examples

The equation

$$x_1x_4(x_2^2+x_3^2+x_5^2)+x_2x_3(x_1^2+x_4^2+x_5^2)=6x_1x_2x_3x_4x_5 \tag{2}$$

admits infinitely many positive integer solutions: $(x_1,x_2,x_3,x_4,x_5) = (1,1,1,1,1)$, $(1,1,1,1,2)$, $(1,5,1,1,3)$, $(29,1,1,5,2)$, $(578,2,1,53,9)$, $(85,2,578,6305,6860)$, ..., etc.

Let $\Gamma_{1,2}$ be the group generated by the rational transformations:

$$\omega_1(x_1,x_2,x_3,x_4,x_5)=(x_4,x_2,x_3,\frac{x_4^2+x_5^2}{x_1},x_5),$$
$$\omega_2(x_1,x_2,x_3,x_4,x_5)=(x_4,x_1,x_2,x_3,\frac{x_1x_3+x_2x_4}{x_5})$$
$$\omega_3(x_1,x_2,x_3,x_4,x_5)=(x_1,\frac{x_2^2+x_5^2}{x_3},x_2,x_4,x_5)$$

If (x_1,x_2,x_3,x_4,x_5) is in the $\Gamma_{1,2}$-orbit of $(1,1,1,1,1)$, then it is a positive integer solution of (2).

Note that ω_1, ω_3, even if they are restricted to the locus of (2), are not polynomial mappings with integer coefficients. For example

$$\omega_1(x_1, x_2, x_3, x_4, x_5) = (x_4, x_2, x_3, 6x_4x_5 - \frac{x_4(x_2^2 + x_3^2 + x_5^2)}{x_2x_3} - x_1, x_5).$$

So the fact that $\Gamma_{1,2}(1, 1, 1, 1, 1)$ are integer solutions is not trivial at this moment.

The following equation is a variant of (2).

$$x_1x_4(x_2^2 + x_3^2 + x_5^2) + x_2x_3(x_1^2 + x_4^2 + x_5^2) = 14x_1x_2x_3x_4x_5. \tag{3}$$

If $(x_1, x_2, x_3, x_4, x_5)$ is in the $\Gamma_{1,2}$-orbit of $(\sqrt{2} + 1, \sqrt{2} - 1, \sqrt{2} + 1, \sqrt{2} - 1, 1)$, then it is a solution of (3) in the ring of integers of the quadratc field $\mathbb{Q}(\sqrt{2})$.

The equation

$$
\begin{aligned}
18x_1x_2x_3x_4x_5x_6x_7x_8x_9 = {} & x_4x_5x_6x_7x_8x_9(x_1^2+x_2^2+x_3^2)+x_1x_5x_6x_7x_8x_9(x_2^2+x_3^2+x_4^2) \\
& +x_2x_3x_5x_6x_7x_8(x_1^2+x_4^2+x_9^2)+x_1x_2x_3x_4x_6x_7(x_5^2+x_8^2+x_9^2) \\
& +x_1x_2x_3x_4x_7x_8(x_5^2+x_6^2+x_7^2)+x_1x_2x_3x_4x_5x_9(x_6^2+x_7^2+x_8^2)
\end{aligned}
$$

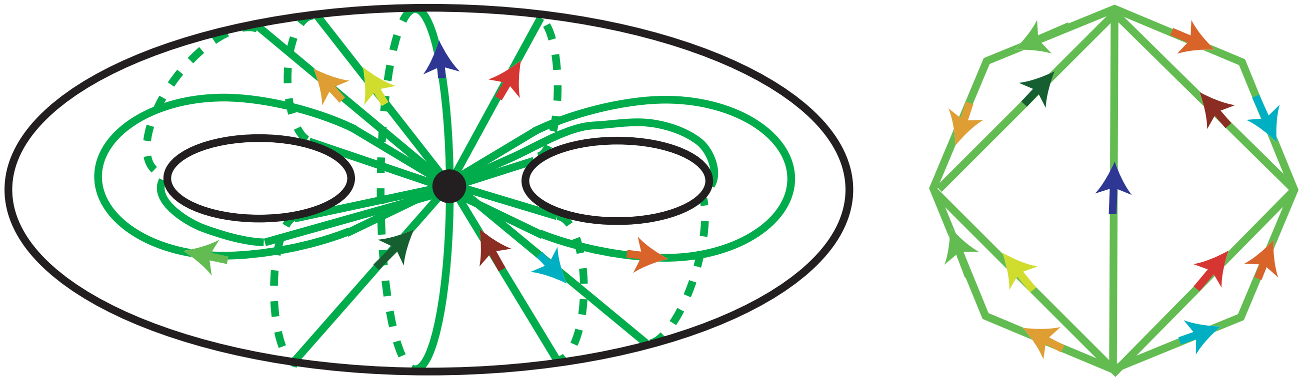
admits infinitely many positive integer solutions: $(1,1,1,1,1,1,1,1,1)$, $(1,1,3,2,1,1,11,1)$, $(4,22,10,2384,691,28,25,1,468)$, $(36,22,134,31752,691,28,25,1,468)$,..., etc.

These solutions are found in the orbit of $(1,1,1,1,1,1,1,1)$ under a group $\Gamma_{2,1}$ of rational transformations. But some of the generators of this group are terribly lengthy to write down.

2 Penner's coordinate-system for the Teichmüller space of a punctured surface

Let $F = F_{g,n}$ be an oriented surface of genus g with n punctures, $n \geq 1$ and $2g-2+n > 0$, and $\mathcal{T}_{g,n}$ denote the Teichmüller space of hyperbolic structures on F with finite area.

Let $\Delta = (c_1, c_2, ..., c_D)$ be an ideal triangulation of F, where $D = 6g-6+3n$.



R.C. Penner introduced a coordinate-system, or a real-analytic embedding

$$\lambda_\Delta : \mathcal{T}_{g,n} \longrightarrow \mathbb{R}^D.$$

Features of Penner's coordinates or λ length coordinates (see Remark 2 in Section 4.1) are:

1. $\lambda_\Delta(\mathcal{T}_{g,n})$ is contained in an affine algebraic variety defined by n polynomials.
2. For two ideal triangulations Δ and Δ', the coordinate change

$$\begin{array}{ccc} \mathcal{T}_{g,n} & \xrightarrow{\lambda_\Delta} & \lambda_\Delta(\mathcal{T}) \subset \mathbb{R}^D \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \lambda_{\Delta'}\circ\lambda_\Delta^{-1}} \\ \mathcal{T}_{g,n} & \xrightarrow{\lambda_{\Delta'}} & \lambda_{\Delta'}(\mathcal{T}) \subset \mathbb{R}^D \end{array}$$

   extends to a *rational transformation* of $\mathbb{R}^D$
3. (Corollary to 2) Let $\mathcal{MC}_{g,n}$ denote the *mapping class group* of F. The correspondence

$$\phi \mapsto \phi_* = \lambda_{\varphi^{-1}(\Delta)} \circ \lambda_\Delta^{-1}$$

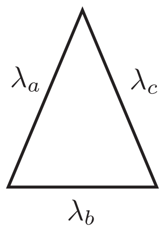
   gives an isomorphism of $\mathcal{MC}_{g,n}$ to a group of rational transformations in $\mathbb{R}^D$.

Remarks

(1) It is easy to obtain the polynomials in the statement 1 (such as the Markoff polynomial for the case of once punctured torus). For the case of once punctured surface, all points in $\lambda_\Delta(\mathcal{T}_{g,1})$ satisfy

$$\sum_T \left(\frac{\lambda_a}{\lambda_b \lambda_c} + \frac{\lambda_b}{\lambda_c \lambda_a} + \frac{\lambda_c}{\lambda_a \lambda_b} \right) - 1 = 0,$$

where a, b, c are edges of a triangle T in Δ and the sum is taken over all triangles T in Δ. To obtain a polynomial we multiply the left hand side by $\prod_{c \in \Delta} \lambda_c$.



(2) The groups $\Gamma_{1,1}$ and $\Gamma_{1,2}$ in Section 1 are the rational representations of $\mathcal{MC}_{1,1}$ and $\mathcal{MC}_{1,2}$, respectively.

3 Integer Solutions of Some Diophantine Equations: revisited

Let $\mathcal{MC}_{g,1}$ be the mapping class group of the once punctured surface of genus $g(>1)$. The rational transformation φ_* for $\varphi \in \mathcal{MC}_{g,1}$ in Penner's coordinates has the form

$$\varphi_*(\lambda_1, ..., \lambda_{d+1}) = \left(\cdots, \frac{P_k(\lambda_1, ..., \lambda_{d+1})}{\lambda_1^{a_1} \cdots \lambda_{d+1}^{a_{d+1}}}, \cdots \right)_{k=1,...,d+1} \tag{4}$$

where $d = 6g - 4 = \dim \mathcal{T}_{g,1}$, and in the k-th entry $P_k(\lambda_1, ..., \lambda_{d+1})$ is a homogeneous polynomial with positive integer coefficients with degree $a_1 + \cdots + a_{d+1} + 1$ and

$$a_i = \text{the geometric intersection number of } \varphi^{-1}(c_k) \text{ and } c_i.$$

The Penner coordinates send $\mathcal{T}_{g,1}$ to an algebraic variety defined by a single polynomial $\Pi(\lambda_1, ..., \lambda_{d+1})$ in $\mathbb{R}^{d+1}$ $(\subset \mathbb{C}^{d+1})$ (see Remark 1 in Section 2).

It is possible to multiply $(\lambda_1, ..., \lambda_{d+1})$ by a suitable constant so that $\Pi(\lambda_1, ..., \lambda_{d+1}) = 0$ admits a solution $(x_1, x_2, ..., x_{d+1})$ with entries in the set of the units in the ring of integers of a number field K.

By the definition of φ_* and (4)

(1) For each mapping class $\varphi \in \mathcal{MC}_{g,1}$, φ_* preserves the locus of $\Pi(\lambda_1, ..., \lambda_{d+1}) = 0$.
(2) $\varphi_*(x_1, x_2, ..., x_{d+1})$ is an integer solution in K.

Remark. The result as above can be generalized to the case of surfaces with more than one puncture to some extent.

4 Complex-valued Penner's coordinates

Let

$$\Gamma = \pi_1(F) = \langle a_1, b_1, ..., a_g, b_g, d_1, ..., d_n : (\prod_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1}) d_1 \cdots d_n = 1 \rangle.$$

and $\mathcal{R}_{g,n}$ denote the space of the classes of faithful representations

$$\rho : \Gamma \to SL(2, \mathbb{C})$$

such that

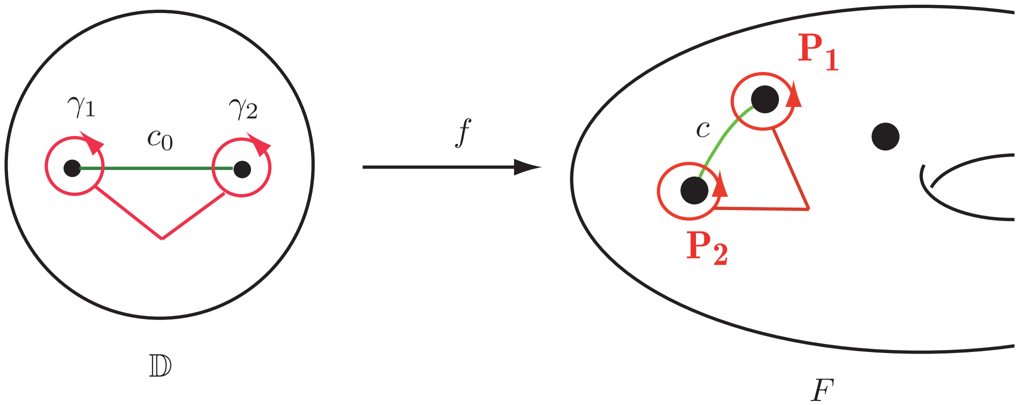
$$\rho(d_k) \text{ is parabolic and } \mathrm{tr}\rho(d_k) = -2 \text{ for } k = 1, 2, ..., n.$$

4.1 Definition of complex λ-length

Let $\mathbb{D} = \{z \in \mathbb{C} : |z - 1/2| \leq 1\} - \{0, 1\}$ be a twice punctured disk which contains the ideal arc $c_0(t) = t$ $(0 < t < 1)$. Then

$$\pi_1(\mathbb{D}) = \langle \gamma_1, \gamma_2 \rangle.$$

Let $f : \mathbb{D} \to F$ be an immersion such that $c = f \circ c_0$ is an ideal arc. For $[\rho] \in \mathcal{R}_{g,n}$, we have two parabolic elements with trace -2: $P_1 = \rho(f_* \gamma_1), P_2 = \rho(f_* \gamma_2)$.



Then two parabolic P_1 and P_2 satisfy

$$(*)\ \mathrm{tr}P_1 = \mathrm{tr}P_2 = -2, \text{ and } P_1 \text{ and } P_2 \text{ do not commute.}$$

There is a $Q \in SL(2, \mathbb{C})$ such that $Q^2 = -P_1 P_2$.

We define

$$\lambda(c, \rho) = \mathrm{tr}Q.$$

Remarks.

(1) $\lambda(c, \rho)$ is defined uniquely up to sign.

(2) If ρ is a Fuchsian representation and $\lambda(c, \rho) > 0$, then $\lambda(c, \rho)$ equals Penner's λ-length up to the constant factor $\sqrt{2}$.

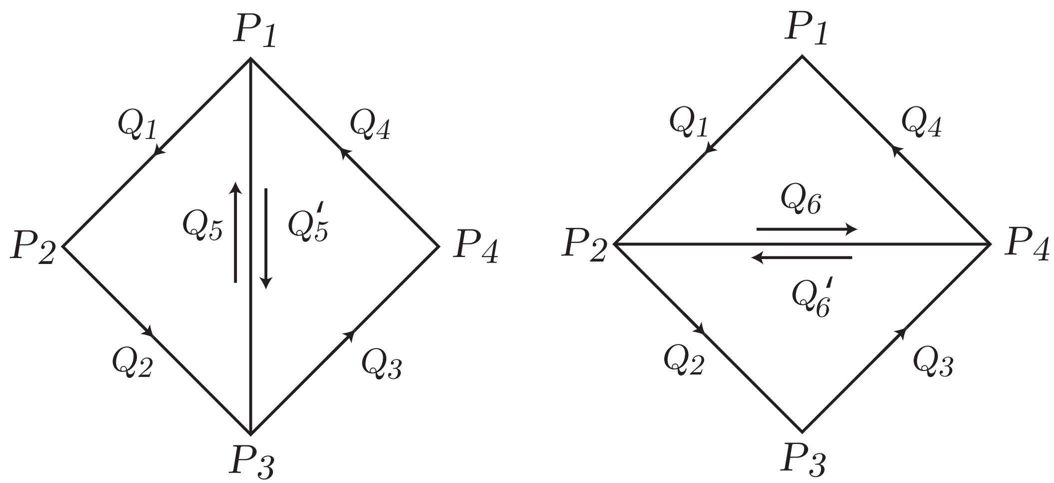
4.2 The ideal Ptolemy identity

We write

$$P_1 \xrightarrow{Q} P_2,$$

if the triple (P_1, P_2, Q) satisfies the condition $(*)$ and $Q^2 = -P_1 P_2$.

Consider the following diagrams:



$$Q'_5 = P_1 Q_5 P_1^{-1},\ Q'_6 = P_4 Q_6 P_4^{-1}.$$

Theorem 1 (the *ideal Ptolemy identity*)

$$\mathrm{tr}Q_5\mathrm{tr}Q_6 = (\pm)\mathrm{tr}Q_1\mathrm{tr}Q_3 + (\pm)\mathrm{tr}Q_2\mathrm{tr}Q_4, \tag{5}$$

where the signs $(\pm)$ depend on the signs of traces of $Q_1Q_2Q_5$,..., $Q_1Q_6Q_4$, which are necessarily parabolic.

Remarks.

(i) The ideal Ptolemy identity proved by Penner is a result of the hyperbolic geometry.

(ii) The identity (5) can be obtained by basic trace relations $\mathrm{tr}A\mathrm{tr}B = \mathrm{tr}AB + \mathrm{tr}AB^{-1}$ and $\mathrm{tr}A = \mathrm{tr}A^{-1}$ in $SL(2,\mathbb{C})$.

The mapping class group $\mathcal{MC}_{g,n}$ can be embedded in the *Ptolemy groupoid* (see Section 7 of Penner's paper [1]) and by this fact $\mathcal{MC}_{g,n}$ is represented as a group of rational transformations.

References

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