Affine deformations of the three-holed sphere and other surfaces

Virginie Charette
Université de Sherbrooke, Canada

Singapore, July 2010

We will look at groups G of affine transformations of $\mathbb{A}^3_1$, affine three-space, whose linear parts are the holonomy of a hyperbolic structure on a surface with boundary.
We will consider the following question: when does G act freely and properly discontinuously on $\mathbb{A}^3_1$?
This is joint work with Todd Drumm and Bill Goldman.

A little history

- Milnor's question: Given a manifold $M = \mathbb{R}^3/G$, where G consists of affine transformations, must G be solvable?
- The alternative to Milnor's question: can a free group of affine transformations act freely and properly discontinuously on $\mathbb{A}^3_1$?
- Margulis' answer (to the alternative): yes - take a Schottky group and add appropriate translational parts to generators.
- Fried-Goldman: If G is not solvable, then taking its linear part embeds it as a (conjugate of a) discrete subgroup of $O(2,1)$.
- Mess: In that case, the linear part of G is not the holonomy of a closed surface.

Notation and terminology

- $\mathbb{R}^3_1 = \mathbb{R}^3$ with symmetric indefinite bilinear form of signature $(2, 1)$:
$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3.$$
- $\mathbb{A}^3_1$ is the affine space modeled on $\mathbb{R}^3_1$
- $\mathbb{H}^2$ denotes the hyperbolic plane; think $\mathbb{H}^2 \subset \mathbb{R}^3_1$.

Cocycles and affine deformations

- **From now on:** $\Gamma < \mathrm{SO}(2,1)^0$ is a free group.
- A cocycle $u \in Z^1(\Gamma, \mathbb{R}^3_1)$ yields a representation $\phi_u : \Gamma \to \mathrm{Aff}(\mathbb{A}^3_1)$, by setting :

$$\phi_u(g) : p \mapsto g(p - o) + u(g),$$

where o is a choice of origin.
- Call $\phi_u(\Gamma)$ an *affine deformation* of Γ, and a *proper affine deformation* if it acts (freely and) properly discontinuously on $\mathbb{A}^3_1$.

The Margulis invariant

The *Margulis invariant* of u, denoted α_u, is the linear functional:

$$\alpha_u(g) = \langle u(g), \mathsf{x}^0(g) \rangle$$

where $\mathsf{x}^0(g)$ is a preferred fixed eigenvector of g.
For hyperbolic g, it is the *signed Lorentzian displacement* along a $\phi_u(g)$-invariant line in $\mathbb{A}^3_1$.

The Margulis invariant and cohomology

For a rank 2 free subgroup $\Gamma = \langle g_1, g_2, g_3 \mid g_3 g_2 g_1 = Id \rangle < \mathrm{SO}(2,1)^0$, $H^1(\Gamma, \mathbb{R}^3_1)$ is parametrized using the Margulis invariant:

$$[u] = (\alpha_u(g_1), \alpha_u(g_2), \alpha_u(g_3))$$

(Drumm-Goldman)

Proper affine deformations lie in an octant of $H^1(\Gamma, \mathbb{R}^3_1)$

Set:

$$\mathcal{H}_g = \{[u] \in H^1(\Gamma, \mathbb{R}^3_1) \mid \alpha_u(g) > 0\}$$

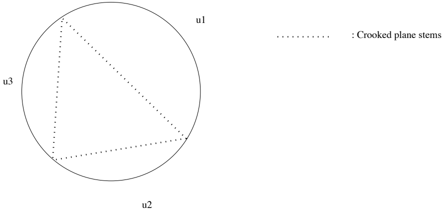
Then the set of proper affine deformations is contained in

$$\bigcap_{g \in \Gamma} \mathcal{H}_g$$

(Margulis+absence of loss of generality)

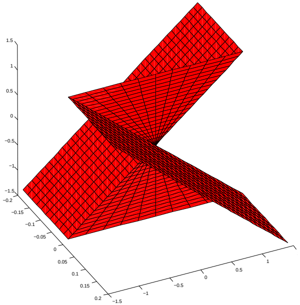
Ideal triangle configurations

The basic idea: Geodesics in $\mathbb{H}^2$ correspond to *crooked planes* in $\mathbb{A}^3_1$.
Our strategy will be to move the crooked planes away from each other along the edges of an ideal triangulation to obtain fundamental domains for proper action



★ We want pairwise disjoint crooked planes in order to apply a Klein-Maskit type combination theorem.

What you need to know about crooked planes for this talk



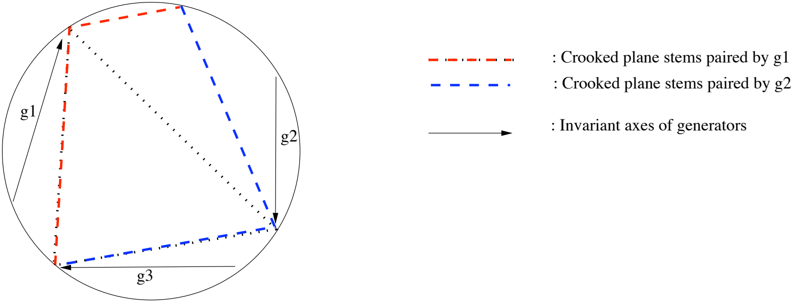
In a world without a positive definite metric, where discrete groups may not act properly, you need something like crooked planes to build fundamental domains.

Ideal triangle configurations and the three-holed sphere

Let $\Sigma = \mathbb{H}^2/\Gamma$ be a three-holed sphere. Write:

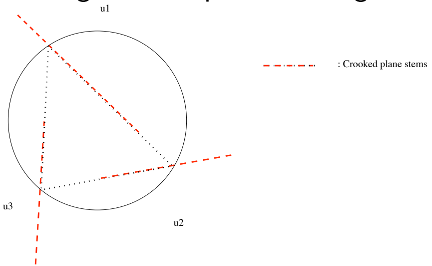
$$\Gamma = \langle g_1, g_2, g_3 \mid g_3 g_2 g_1 = Id \rangle$$

Let T be the ideal triangle whose vertices are the attracting fixed points of the g_i.

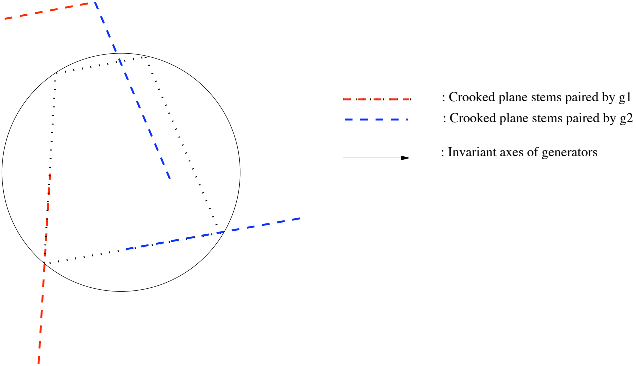


T yields half of an ideal triangulation of the interior of Σ's convex core.

With a little work, one obtains proper affine deformations by moving crooked planes "along the edges of T".



Such a move will look something like this for a quadruple of crooked planes.



For the three-holed sphere, this yields all of $\mathcal{H}_{g_1} \cap \mathcal{H}_{g_2} \cap \mathcal{H}_{g_3}$

Theorem (C-Drumm-Goldman)

Let $\Gamma = \langle g_1, g_2, g_3 \mid g_3 g_2 g_1 = Id \rangle < \mathrm{SO}(2,1)^0$, *which is the holonomy of a three-holed sphere, and let* $u \in Z^1(\Gamma, \mathbb{R}^3_1)$*; suppose that* $\alpha_u(g_i)$, $i = 1, 2, 3$, *are all of the same sign. Then* $\phi_u(\Gamma)$ *admits a fundamental domain. In particular, it acts freely and properly discontinuously on* $\mathbb{A}^3_1$.

(Compare Jones and Goldman-Labourie-Margulis.)

Ideal triangle configurations for the one-holed torus

Let $\Gamma = \langle g_1, g_2, g_3 \mid g_1 g_2 g_3 = Id \rangle$ now be the holonomy of a one-holed torus.
We consider an ideal triangulation with vertices judiciously chosen amongst the fixed points of the commutators.
We obtain various regions of proper cocycles by *changing the generating set*:

$$(g_1, g_2, g_3) \mapsto (g_2^{-1}, g_1, g_1^{-1} g_2)$$

Result for the one-holed torus

Theorem (C-Drumm-Goldman)

Let Γ be the holonomy of a one-holed torus. The set of cohomology classes of cocycles admitting a fundamental domain is the **interior** *of the set:*

$$\bigcap_{g \in SCC} \mathcal{H}_g$$

where $SCC \subset \Gamma$ is the set of elements corresponding to simple closed curves.