Character varieties and Morse theory

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Outline

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Cohomology of Kähler and hyperKähler quotients

- The goal is to compute the equivariant cohomology of symplectic (Kähler or hyperKähler) reductions.
- By the Kempf-Ness, Guillemin-Sternberg theorem, examples arise in geometric invariant theory.
- Kirwan, Atiyah-Bott: In the symplectic case there is a “perfect" Morse stratification.
- HyperKähler case still unknown.

Infinite dimensional examples:

- Higgs bundles (Hitchin)
- Stable pairs (Bradlow)
- Quiver varieties (Nakajima)

These involve symplectic reduction in the presence of singularities.

Key points:

- this poses no (additional) analytic difficulties.
- Singularities can cause the Morse stratification to lose “perfection."
- Computations of cohomology are (sometimes) still possible.

Application to representation varieties

- $M =$ a closed Riemann surface $g \geq 2$
- $\pi = \pi_1(M, *)$
- $G =$ a compact connected Lie group
- $G^{\mathbb{C}} =$ its complexification (e.g. $G = U(n)$, $G^{\mathbb{C}} = GL(n, \mathbb{C})$)
- Representation varieties:

$$\mathrm{Hom}(\pi, G)/G \qquad vs. \qquad \mathrm{Hom}(\pi, G^{\mathbb{C}})/\!\!/G^{\mathbb{C}}$$

$$\wr| \qquad\qquad\qquad\qquad \wr|$$

Moduli of G-bundles   Moduli of G-Higgs bundles

Equivariant cohomology

- Z topological space with an action by G
- Classifying space: $EG \to BG$ is a contractible, principal G-bundle
- Equivariant cohomology: $H^*_G(Z) = H^*(Z \times_G EG)$
- If the action is free: $H^*_G(Z) = H^*(Z/G)$
- If the action is trivial: $H^*_G(Z) = H^*(Z) \otimes H^*(BG)$

Application to representation varieties

Theorem (DWWW)

The equivariant Poincaré polynomial is given by

$$
\begin{aligned}
P_t^{SL(2,\mathbb{C})}(\mathrm{Hom}(\pi, SL(2,\mathbb{C}))) &= \frac{(1+t^3)^{2g} - (1+t)^{2g} t^{2g+2}}{(1-t^2)(1-t^4)} \\
&- t^{4g-4} + \frac{t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} + \frac{(1-t)^{2g} t^{4g-4}}{4(1+t^2)} \\
&+ \frac{(1+t)^{2g} t^{4g-4}}{2(1-t^2)} \left(\frac{2g}{t+1} + \frac{1}{t^2-1} - \frac{1}{2} + (3-2g) \right) \\
&+ \tfrac{1}{2}(2^{2g}-1) t^{4g-4} \left((1+t)^{2g-2} + (1-t)^{2g-2} - 2 \right)
\end{aligned}
$$

Application to representation varieties

- The Torelli group $I(M)$ is the subgroup of the mapping class group that acts trivially on the homology of M.

Theorem

$I(M)$ acts nontrivially on the equivariant cohomology of $\mathrm{Hom}(\pi, SL(2,\mathbb{C}))$ (via Prym representations).

- By contrast, $I(M)$ always acts trivially on the equivariant cohomology of $\mathrm{Hom}(\pi, G)$, G compact.

Outline

Introduction

Higgs bundles

Equivariant Morse theory

Prym representations

Higgs bundles

- $\mathcal{A}$ = space of unitary connections on a trivial rank 2 hermitian vector bundle $E \to M$
- $\mathrm{ad}\, E$ = bundle of skew-hermitian endomorphisms of E
- $\mathcal{A} \simeq \Omega^1(M, \mathrm{ad}\, E)$
- $\mathcal{A}_c = \left\{(A, \Psi) : A \in \mathcal{A}\ ,\ \Psi \in \Omega^1(M, \sqrt{-1}\, \mathrm{ad}\, E)\right\}$
- $\mathcal{G}$ = group of unitary gauge transformations, $\mathcal{G}^{\mathbb{C}}$ its complexification.
- Action of $\mathcal{G}$ on $\mathcal{A}_c$ is hamiltonian with respect to *three* symplectic structures.

The moduli space

The three moment maps for the action of $\mathcal{G}$ are:

$$\mu_1(A, \Psi) = F_A + \frac{1}{2}[\Psi, \Psi]$$
$$\mu_2(A, \Psi) = \sqrt{-1}\, d_A \Psi$$
$$\mu_3(A, \Psi) = \sqrt{-1}\, d_A(*\Psi)$$

We are interested in the common zero locus:

$$\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0) / \mathcal{G}$$

Hitchin's ASD equations

- The **Higgs bundles** are

$$\mu_2^{-1}(0) \cap \mu_3^{-1}(0) = \left\{ (A, \Phi) : \bar{\partial}_A \Phi = 0 \right\} = \mathcal{B}_{higgs}$$

  where $\Psi = \Phi + \Phi^*$, $\Phi \in \Omega^{1,0}(M, \mathrm{End}\, E)$
- The conditions

$$\mu_1(A, \Phi) = F_A + [\Phi, \Phi^*] = 0$$

  are called **Hitchin's ASD equations.**

Flat connections

- The **flat connections** are: $\mu_1^{-1}(0) \cap \mu_2^{-1}(0) = \mathcal{A}_c^{flat}$

  $= \{(A, \Psi) \in \mathcal{A}_c : D = d_A + \Psi \text{ is flat } \mathrm{SL}(2, \mathbb{C}) \text{ connection}\}$

- Given a flat connection D, $\mathrm{hol}(D) = \rho : \pi \to \mathrm{SL}(2, \mathbb{C})$, there is a ρ-equivariant map

$$u : \widetilde{M} \to \mathbb{H}^3 \simeq \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$$

- The equation

$$\mu_3(A, \Psi) = \sqrt{-1}\, d_A(*\Psi) = 0$$

  is equivalent to u being **harmonic.**

Theorem (Hitchin, Simpson)

The closure of the $\mathcal{G}^{\mathbb{C}}$-orbit of a Higgs bundle (A, Φ) intersects $\mu_1^{-1}(0)$ if and only if $(A, \Phi) \in \mathcal{B}^{ss}_{higgs}$, the subset of semistable Higgs bundles.

Theorem (Corlette, Donaldson)

If $\rho \in \mathrm{Hom}(\pi, \mathsf{SL}(2, \mathbb{C}))$, then the closure of its $\mathsf{SL}(2, \mathbb{C})$ orbit intersects $\mu_3^{-1}(0)$.

Corollary

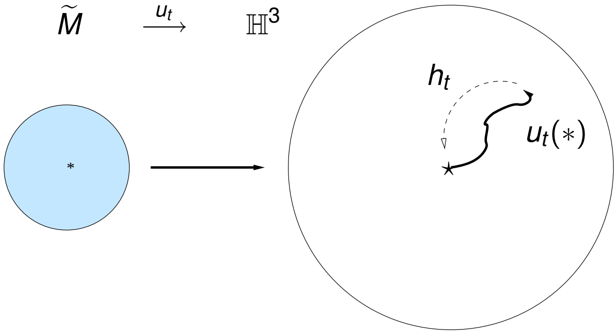
The **moduli space of semistable Higgs bundles is**

$$\begin{aligned} \mathfrak{M}_{higgs} &= \mathcal{B}^{ss}_{higgs} /\!\!/ \mathcal{G}^{\mathbb{C}} \\ &\simeq \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0) / \mathcal{G} \\ &\simeq \mathrm{Hom}(\pi, \mathsf{SL}(2, \mathbb{C})) /\!\!/ \mathsf{SL}(2, \mathbb{C}) \end{aligned}$$

Deformation retraction for the character variety

Fix ρ and consider ρ-equivariant maps

$$\widetilde{M} \xrightarrow{u_t} \mathbb{H}^3$$



- $\partial u_t/\partial t = -\tau(u_t)$ is the ρ-equivariant harmonic map flow
- $h_t u_t(*) = \star$, fixed.
- Define a flow on $\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$ by $\rho_t = h_t \rho h_t^{-1}$

Deformation retraction for the character variety

$$r : \mathcal{A}_c^{flat} \longrightarrow \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0) : \rho \mapsto \lim_{t \to \infty} \rho_t$$

Theorem

The map r defines a $\mathcal{G}$-equivariant deformation retraction

$$\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0) \hookrightarrow \mathrm{Hom}(\pi, \mathsf{SL}(2, \mathbb{C}))$$

Deformation retraction on Higgs bundles

Study the gradient flow on $\mathcal{B}_{higgs}$ of the Yang-Mills-Higgs functional:

$$\mathrm{YMH}(A, \Phi) = \| F_A + [\Phi, \Phi^*] \|^2$$

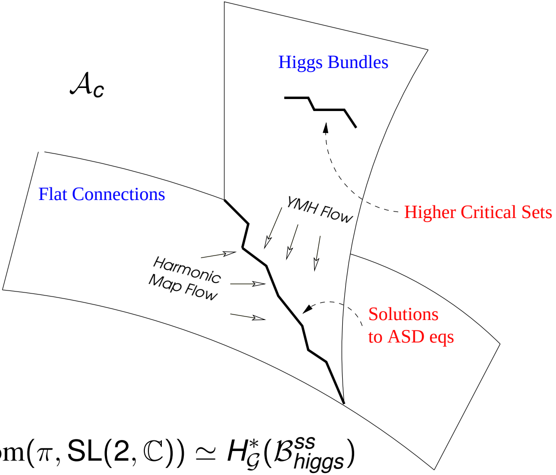
Theorem (Wilkin)

The gradient flow converges (to the expected limit). In particular, the inclusion

$$\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0) \hookrightarrow \mathcal{B}_{higgs}^{ss}$$

is a $\mathcal{G}$-equivariant deformation retraction.

Higgs Bundles vs. Flat Connections



$$H^*_{\mathsf{SL}(2,\mathbb{C})}(\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C})) \simeq H^*_{\mathcal{G}}(\mathcal{B}^{ss}_{higgs})$$

Approach: Compute the right hand side via Morse theory.

Outline

Symplectic reduction

- (X, ω) symplectic manifold (compact)
- G compact, connected Lie group, acting symplectically.
- $\mu : X \to \mathfrak{g}^*$ a moment map ($d\mu^\xi(\cdot) = \omega(\xi^\sharp, \cdot)$)
- the Marsden-Weinstein quotient $\mu^{-1}(0)/G$ is symplectic.
- What is the cohomology of $\mu^{-1}(0)/G$?

Atiyah-Bott, Kirwan

- Study the gradient flow $f = \|\mu\|^2$.
- Critical sets η_β are characterized in terms of isotropy in G.
- Gradient flow $\rightsquigarrow$ **smooth** stratification $X = \cup_{\beta \in I} S_\beta$; with normal bundles ν_β.
- The corresponding long exact sequence **splits**

$$\cdots \longrightarrow H_G^*(S_\beta, \cup_{\alpha<\beta} S_\alpha) \longrightarrow H_G^*(S_\beta) \longrightarrow H_G^*(\cup_{\alpha<\beta} S_\alpha) \longrightarrow \cdots$$

- Compute change at each step from $H_G^*(S_\beta, \cup_{\alpha<\beta} S_\alpha)$

Two key steps

- **Morse-Bott Lemma:**

$$H^*_G(S_\beta, \cup_{\alpha<\beta} S_\alpha) \simeq H^*_G(\nu_\beta, \nu_\beta \setminus \{0\}) \simeq H_G^{*-\lambda_\beta}(\eta_\beta)$$

- **Atiyah-Bott Lemma:** criterion for multiplication by the equivariant Euler class to be injective.

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^p_G(S_\beta, \cup_{\alpha<\beta} S_\alpha) & \longrightarrow & H^p_G(S_\beta) & \longrightarrow \cdots \\ & \downarrow \cong & & \downarrow & \\ & H^p_G(\nu_\beta, \nu_\beta \setminus \{0\}) & \longrightarrow & H^p_G(\eta_\beta) & \end{array}$$

Perfect equivariant Morse theory

Theorem (Kirwan, Atiyah-Bott)

For a (compact) symplectic manifold X with action by G,

$$P_t^G(\mu^{-1}(0)) = P_t^G(X) - \sum_{\beta} t^{\lambda_\beta} P_t^G(\eta_\beta)$$

Theorem (Kirwan surjectivity)

The map on cohomology

$$H_G^*(X) \longrightarrow H_G^*(\mu^{-1}(0))$$

induced from inclusion $\mu^{-1}(0) \hookrightarrow X$ is **surjective**.

Vector bundles on Riemann surfaces

- $\mu : \mathcal{A} \to Lie(\mathcal{G})$ is given by $A \mapsto F_A$
- Minimum of $\|\mu\|^2 = \|F_A\|^2 = YM(A)$ is the space of flat connections (i.e. representation variety)
- The flow converges and the Morse stratification is smooth (Daskalopoulos)
- Higher critical sets correspond to split Yang-Mills connections, i.e. representations to smaller groups. For example, $E = L_1 \oplus L_2$, $d = \deg L_1 > \deg L_2$:

$$\eta_d = Jac(M) \times Jac(M)$$

- Morse-Bott lemma: Negative directions given by $H^{0,1}(L_1^* \otimes L_2)$; $\lambda_d = \dim$ is constant.

Theorem (Atiyah-Bott, Daskalopoulos)

$$P_t^{SU(2)}(\mathrm{Hom}(\pi, SU(2))) = P(B\mathcal{G}) - \sum_{d=0}^{\infty} t^{\lambda_d} P_t^{S^1}(Jac_d(M))$$
$$= \frac{(1+t^3)^{2g} - t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)}$$

Corollary

Kirwan surjectivity: $H^*(B\mathcal{G}) \to H^*_{SU(2)}(\mathrm{Hom}(\pi, SU(2)))$ *is surjective. In particular,* $\mathcal{I}(S)$ *acts trivially on the* SU(2)*-equivariant cohomology of* $\mathrm{Hom}(\pi, \mathrm{SU}(2))$.

Singularities

- What about Higgs bundles (A, Φ)?
- Singularities because of the jump in $\dim \ker \bar{\partial}_A$.
- Kuranishi model: $\{\text{Slice}\} \hookrightarrow H^1(\text{deformation complex})$
- Negative directions: ν_β is the intersection of negative directions with the image of the slice.
- Morse-Bott isomorphism: Need to define a deformation retraction.

Critical Higgs bundle

- ν_d: $E = L_1 \oplus L_2$, $d = \deg L_1 > \deg L_2$,

$$A = A_1 \oplus A_2 \quad , \quad \Phi = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix}$$

- Negative directions ν_d: (a, φ) strictly lower triangular.

$$a \in H^{0,1}(L_1^* \otimes L_2) \, , \, \varphi \in H^{1,0}(L_1^* \otimes L_2)$$

- $\deg(L_1^* \otimes L_2) < 0 \;\Rightarrow \dim H^{0,1}(L_1^* \otimes L_2)$ constant.
- $\deg(L_1^* \otimes L_2 \otimes K_M)$ is not necessarily negative, so $\dim H^{1,0}(L_1^* \otimes L_2)$ can jump.

- Can still prove $H^*_{\mathcal{G}}(X_d, X_{d-1}) \simeq H^*_{\mathcal{G}}(\nu_d, \nu_d \setminus \{0\})$
- But the exact sequence

$$\cdots \longrightarrow H^*_G(X_d, X_{d-1}) \longrightarrow H^*_G(X_d) \longrightarrow H^*_G(X_{d-1}) \longrightarrow \cdots$$

does **not** split in general.

Theorem

For the case of Higgs bundles, Kirwan surjectivity **holds** *for* $\mathrm{GL}(2, \mathbb{C})$ *but* **fails** *for* $\mathrm{SL}(2, \mathbb{C})$*.*

And in fact, the Torelli group acts nontrivially...

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Prym representations

- $\Gamma_2 = H^1(M, \mathbb{Z}/2) \simeq \mathrm{Hom}(\pi, \{\pm 1\})$
- Γ_2 acts on $\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$ by $(\gamma\rho)(x) = \gamma(x)\rho(x)$
- This action commutes with conjugation by $\mathsf{SL}(2,\mathbb{C})$, and hence it defines an action on the ordinary and equivariant cohomologies.

Prym representations

- $1 \neq \gamma \in \Gamma_2$, defines an unramified double cover $M_\gamma \to M$
- $H^1(M_\gamma) = W_\gamma^+ \oplus W_\gamma^-$ (± 1 eigenspaces of the involution)
- Lifts of elements of $\mathcal{I}(M)$ that commute with the involution may or may not be in the Torelli group of M_γ.
- Hence, there is a representation

  $$\Pi_\gamma : \mathcal{I}(M) \longrightarrow \mathrm{Sp}(W_\gamma^-, \mathbb{Z})/\{\pm I\}$$

  called the *Prym representation* of $\mathcal{I}(M)$ associated to γ.
- The image of Π_γ has finite index for $g > 2$ (Looijenga).

Action of the Torelli group

Set: $H^*_{eq.} = H^*_{\mathsf{SL}(2,\mathbb{C})}(\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C})))$

Theorem (DWW)

1. *$\mathcal{I}(M)$ acts trivially on $(H^*_{eq.})^{\Gamma_2}$*
2. *For $q \in S = \{2j\}_{j=1}^{g-2}$ the action of $\mathcal{I}(M)$ splits as*

$$H^{6g-6-q}_{eq.} = (H^{6g-6-q}_{eq.})^{\Gamma_2} \oplus \bigoplus_{1 \neq \gamma \in \Gamma_2} \Lambda^q W^-_\gamma$$

3. *$\mathcal{I}(M)$ acts trivially on $H^{6g-6-q}_{eq.}$ for $q \notin S$*