On primitive stable representations of geometrically infinite handlebody hyperbolic 3-manifolds

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Basic notions of hyperbolic 3-manifolds

- $\mathbb{H}^3$ is the upper half space in $\mathbb{R}^3$ with a point at ∞.
- The boundary of $\mathbb{H}^3$ can be identified with $\hat{\mathbb{C}}$.
- Aut($\hat{\mathbb{C}}$) =PSL(2,$\mathbb{C}$)= The group of orientation preserving isometries of $\mathbb{H}^3$ with respect to the hyperbolic metric $ds^2 = \frac{|dz|^2+dt^2}{t^2}$.
- *A Kleinian group* is defined as a discrete subgroup of PSL(2,$\mathbb{C}$).
- *The limit set* L_Γ of a Kleinian group Γ is defined as the set of limit points of the orbit Γx in $\hat{\mathbb{C}}$ for some $x \in \mathbb{H}^3 \cup \hat{\mathbb{C}}$.
- A 3-manifold M is called a *complete hyperbolic 3-manifold* if it can be represented as $\mathbb{H}^3/\Gamma$ for a Kleinian group Γ.

I will assume that Γ does not contain any parabolic or elliptic isometries.

Compact core and convex core

- *A compact core* is a compact submanifold of M such that its inclusion is a homotopy equivalence.
- *The convex core* C_Γ of $M = \mathbb{H}^3/\Gamma$ is the smallest convex submanifold homotopically equivalent to M
- C_Γ can be constructed as a quotient of convex hull of L_Γ by Γ.

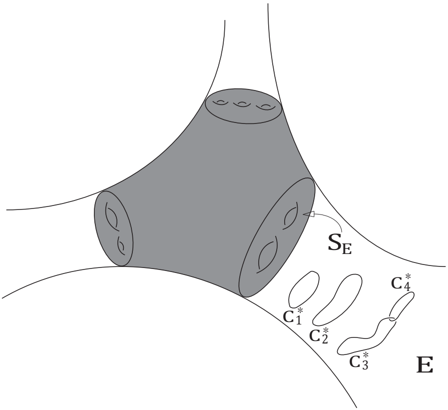

Figure: The dark region describes a compact core of M.

Geometrically finite and infinite hyperbolic 3-manifolds

- M is *geometrically finite* $\Leftrightarrow \mathbb{C}_\Gamma$ is compact $\Leftrightarrow$ Every end E of M has a neighborhood disjoint with $\mathbb{C}_\Gamma$.
- Every geometrically finite end E corresponds to a Riemann surface in $(\hat{\mathbb{C}} \backslash L_\Gamma)/\Gamma$ which defines a point in Teichmüller space $\mathcal{T}(S_E)$.
- For a geometrically infinite end E, we have an ending lamination λ_E on S_E instead of a point in $\mathcal{T}(S_E)$.
- *An ending lamination* is roughly a limit of simple closed geodesics exiting E.
- Ending lamination theorem says the end invariants completely determine the geometry of hyperbolic 3-manifolds(Brock, Canary, Minsky).

Schottky groups

Choose $2n$ disjoint round disks $\{D_1, D'_1, \ldots, D_n, D'_n\}$ on $\hat{\mathbb{C}}$ and loxodromic isometries $\{g_1, \ldots, g_n\}$ s.t. $g_i(D_i) = \overline{D_i'^c}$.

- $\langle g_1, \ldots, g_n \rangle$ becomes a rank n free discrete subgroup of PSL(2,$\mathbb{C}$) called *a classical Schottky group*.
- Every geometrically finite free Kleinian group is Schottky.
- For a geometrically infinite free Γ without parabolics, $\mathbb{H}^3/\Gamma = H \cup (\partial H \times [0, \infty))$ by the Tameness theorem and an ending lamination λ is defined on ∂H.

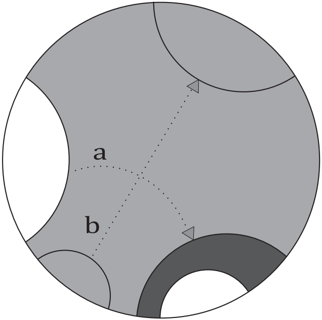

Figure: The grey region maps to the dark region by a. This figure describes the Schottky group with 2-generators $\{a, b\}$.

Geodesic and measured lamiations

- *A geodesic lamination* λ on a closed hyperbolic surface S is a closed subset which is a disjoint union of simple, complete geodesics called leaves of λ.
- A geodesic lamination λ is called *minimal* if every leaf is dense in λ.
- A geodesic lamination λ with a transverse invariant measure μ is called a *measured lamination*.

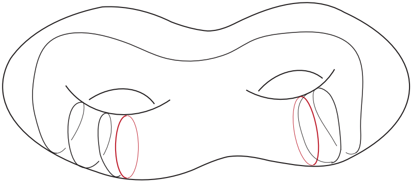

Figure: A geodesic lamination with a spiraling geodesic.

Filling laminations

- A measured lamination is called *filling* if every simple closed geodesic has a nontrivial intersection with it.
- Every filling lamination (λ, μ) has only one minimal component and its complementary regions are ideal polygons.
- An ending lamination λ_E is defined as the support of a limit in $\mathcal{ML}(S_E)$ of $\{c_n\}$ where its geodesic representative c_n^* exits the end E, it is well defined by the intersection numer lemma(Bonahon, Canary).
- Every ending lamination is filling and belongs to Masur domain for compressible ends(Thurston, Canary).
- Originally Masur domain is a set of measured laminations which intersect compressible curves and their limits nontivially, but we can see that this definition is independant of the transverse invariant measure chosen.

Geometric decomposition of PSL(2,$\mathbb{C}$)-characters

Let F_n be a nonabeilian free group on $n \geq 2$ generators. For any group G, Aut(F_n) acts on Hom(F_n, G) by $\alpha(\varphi) = \varphi \circ \alpha^{-1}$ where $\alpha \in$ Aut(F_n), $\varphi \in$Hom(F_n, G).

- G =the noncompact Lie group PSL(2,$\mathbb{C}$), $X_n(G)$ =Hom(F_n, G)/Inn(G).
- The decomposition $X_n(G) = \mathcal{D}(F_n) \cup \mathcal{E}(F_n)$ is invariant under the action of Out(F_n).
- $\mathcal{D}(F_n)$ =discrete faithful characters, $\mathcal{E}(F_n)$ =characters with dense image.
- Schotty characters $\mathcal{S}(F_n)$ is the interior of $\mathcal{D}(F_n)$ and Aut(F_n) acts on $\mathcal{S}(F_n)$ properly discontinuously(Sullivan).

Dynamical decomposition of PSL(2,$\mathbb{C}$)-characters

Minsky and Lubotzky introduced another decomposition of $X_n(G)$ by *primitive stable* and *redundent* charaters.

- $X_n(G) \supset \mathcal{PS}(F_n) \cup \mathcal{R}(F_n)$, it is not known whether $\mathcal{PS}(F_n) \cup \mathcal{R}(F_n)$ is conull in $X_n(G)$ or not.
- Out(F_n) acts ergodically on $\mathcal{R}(F_n)$ and acts properly discontinuously on $\mathcal{PS}(F_n)$(Gelander, Minsky).
- Minsky showed $\mathcal{PS}(F_n)$ is open and is strictly larger than $\mathcal{S}(F_n)$.
- Minsky's conjecture : Every geometrically infinite free representation without parabolics is primitive stable.

Primitive stability-1

- Let w be a primitive and cyclically reduced word in F_n. $\widetilde{w}$ is defined as in the figure.
- Given a representation $\rho : F \to \mathrm{PSL}(2,\mathbb{C})$ and a fixed base point o in $\mathbb{H}^3$, define $\tau_{\rho,o} : \widetilde{\vee S^1} \to \mathbb{H}^3$ as the unique ρ-equivariant map sending the origin e of the universal tree to o and sending each edge to a geodesic segment.

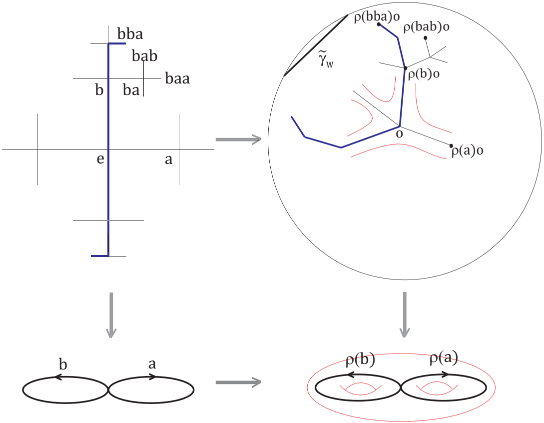

Figure: $\tau_{o,\rho}(\widetilde{w})$ for $w = bba\cdots$

Primitive stability-2

- Let $\rho : F \to$PSL(2,$\mathbb{C}$) be a Schottky representation. Then for any cyclically reduced word w, $\widetilde{\gamma}_w$ belongs to a C-neighborhood of $\tau_{\rho,o}(\widetilde{w})$ where C do not depend on w.
- A representation $\rho : F \to$PSL(2,$\mathbb{C}$) is called *primitive stable* if for any primitive cyclically reduced word w, $\tau_{\rho,o}(\widetilde{w})$ is a (K, δ)-quasi geodesic for uniform constants K, δ. Thus Schottky $\Rightarrow$ Primitive stable.
- Key Lemma : If $\rho : F \to$PSL(2,$\mathbb{C}$) is a discrete faithful representation which is geometrically infinite without parabolics and is not primitive stable, we can find a sequence of primitive cyclically reduced words $\{w_n\}$ such that $d(o, \widetilde{\gamma}_{w_n}) \to \infty$ as $n \to \infty$.

Sphere filling peano curve

Let's describe the original argument of Cannon and Thurston.

- Let S be a closed orientable hyperbolic surface and $\phi : S \to S$ be a pseudo-anosov diffeomorphism.
- The mapping torus $M = M(\phi) = (S \times [0, 1]) / \{(x, 0) \sim (\phi(x), 0)\}$ has a hyperbolic structure by Thurston.
- Let M_S be the infinite cyclic covering space of M such that its fundamental group is isomorphic to $\pi_1(S)$. M_S becomes a double degenerate hyperbolic 3-manifold.
- The homotopy equivalence $i : S \to M_S$ can be lifted so that it induces a continuous embedding $\widetilde{i} : \mathbb{H}^2 \to \mathbb{H}^3$.
- The continuous extension $\widehat{i}$ of $\widetilde{i}$ to $S^1_\infty = \partial_\infty \mathbb{H}^2$ is called the Cannon-Thurston map for M and $\widehat{i}(S^1_\infty)$ fills the sphere $\hat{\mathbb{C}}$.

Identified points of Cannon-Thurston map

For Hyperbolic metric spaces X, Y and an embedding $i : Y \to X$, A Cannon-Thurston map $\widehat{i} : \widehat{Y} \to \widehat{X}$ is a continuous extension of i to their Gromov boundaries. Let $\rho : F_n \to$PSL(2,$\mathbb{C}$) be a discrete faithful representation which is geometrically infinite without parabolics.

- Tameness implies $\mathbb{H}^3/\rho(F_n) = H \cup (\partial H \times [0, \infty))$. Let $S_E = \partial H$.
- Let $\widehat{i} : F_n \cup \partial_\infty F_n \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ be the ρ-equivariant CT-map. For $a, b \in \partial_\infty F_n$, suppose that $\hat{i}(a) = \hat{i}(b)$ and let γ_∞ be the unique biinfinite geodesic on S_E which is homotopic to the biinfinite line in the tree with end points a, b. Then γ_∞ is either a leaf of λ_E or an isolated geodesic joining two ideal boundary points of a complementary polygon of λ_E(Cannon, Thurston, Minsky, Mcmullen, Bowditch, Souto, Mj,...).

Whitehead lemma-1

Fix a generating set $X = \{x_1, x_2, ..., x_n\}$ for F_n. For a cyclically reduced primitive word g, $Wh(g, X)$ is a graph with $2n$ vertices labeled $x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}$ and two vertices x, y^{-1} is joined by an edge from x to y^{-1} if the string xy appears in g or in a cyclic permutation of g.

- If $Wh(g, X)$ is connected and and has no cutpoint, then g is not primitive(Whitehead lemma).
- $Wh(\lambda_E, \Delta)$ can be defined for an ending lamination λ_E on ∂H with a disk system Δ dual to a generating set of F_n.
- For an ending lamination λ_E, there exist a Δ s.t. $Wh(\lambda_E, \Delta)$ is connected and has no cut point(Canary, Otal).

Whitehead lemma-2

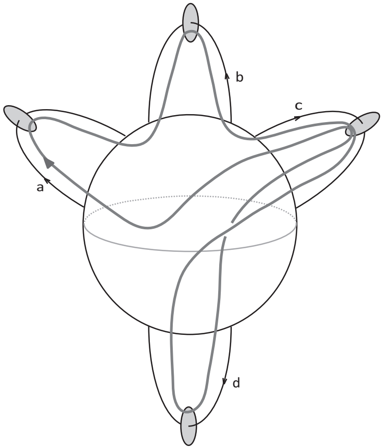

Figure: A loop representing $ab^{-1}cd^{-1}c^{-1}$

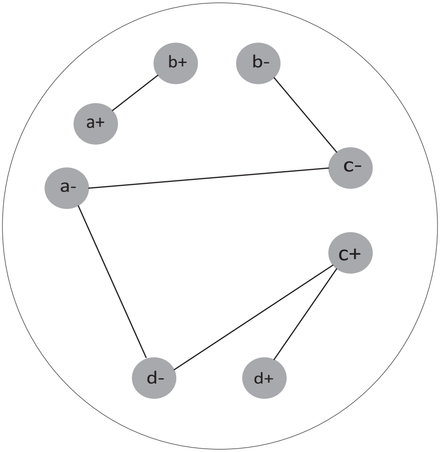

Figure: The Whitehead graph for $ab^{-1}cd^{-1}c^{-1}$

Proof of the main theorem

Theorem : Every discrete faithful representation $\rho : F_n \to \mathrm{PSL}(2,\mathbb{C})$ without parabolics is primitive stable.
Proof :

- Suppose that ρ is not primitive stable. Then we get a sequence of primitive cyclically reduced words $\{w_n\}$ s.t. $d(o, \widetilde{\gamma}_{w_n}) \to \infty$.
- By a variant of Cantor diagonal argument, we can assume that for all $i > 0$, $w_{i+1} = w_i v_i$ for some word v_i.
- The two end points $a, b \in \partial_\infty(F_n)$ of $\widetilde{w}_\infty$ are identified by $\widehat{i}$.
- w_∞ can be straightened to a bi-infinite geodesic γ_∞ on ∂H and its closure contains $\lambda_E \Rightarrow Wh(\gamma_\infty, \Delta)$ is connected and has no cut point.
- For a sufficiently large n, $Wh(w_n, \Delta) = Wh(\gamma_\infty, \Delta)$ but this contradicts to the Whitehead lemma.

Extension to the parabolic cases

A generalized version of Minsky's conjecture is the following : A discrete faithful representation of F_n is primitive stable if and only if every component of its ending lamination is blocking.

- An incompressible component E_i should be geometrically finite because if not, there exists a sequence of exiting primitive closed geodesics.
- λ is called *blocking* with respect to Δ if λ has no Δ-waves, and there exists some k such that every length k subword of the infinite word determined by a leaf of λ does not appear in a cyclically reduced primitive word.
- Doubly incompressible laminations satisfy the blocking property.
- A measured lamination λ is called *doubly incompressible* if for any essential disc or annulus A, $i(\partial A, \lambda) > 0$.
- Under the additional assumption that every compressible component of its ending lamination is doubly incompressible, this conjecture can be proved.

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