Delambre-Gauss Formulas in Hyperbolic 4-Space

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1. Trigonometric formulas for spherical and hyperbolic triangles

Spherical triangles. Consider a spherical triangle in the unit sphere having side-lengths $a, b, c \in (0, \pi)$ and corresponding opposite interior angles $\alpha, \beta, \gamma \in (0, \pi)$.

The following Delambre-Gauss formulas were discovered by Delambre in 1807 (published in 1809) and were subsequently discovered independently by Gauss.

Delambre (1749–1822): Director of Paris Observatory during 1804–1822.
Gauss (1777–1855): Director of Göttingen Observatory during 1807–1855.

Theorem 1.1 (Delambre-Gauss formulas for spherical triangles).

$$\cos\tfrac{1}{2}(a+b)\,\sin\tfrac{1}{2}\gamma = \cos\tfrac{1}{2}(\alpha+\beta)\,\cos\tfrac{1}{2}c, \tag{1}$$

$$\sin\tfrac{1}{2}(a+b)\,\sin\tfrac{1}{2}\gamma = \cos\tfrac{1}{2}(\alpha-\beta)\,\sin\tfrac{1}{2}c, \tag{2}$$

$$\cos\tfrac{1}{2}(a-b)\,\cos\tfrac{1}{2}\gamma = \sin\tfrac{1}{2}(\alpha+\beta)\,\cos\tfrac{1}{2}c, \tag{3}$$

$$\sin\tfrac{1}{2}(a-b)\,\cos\tfrac{1}{2}\gamma = \sin\tfrac{1}{2}(\alpha-\beta)\,\sin\tfrac{1}{2}c. \tag{4}$$

Remark. Note that $a > b$ iff $\alpha > \beta$, and $a + b > \pi$ iff $\alpha + \beta > \pi$.

Corollary 1.2 (Napier's analogies for spherical triangles).

$$\frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)} = \frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c}, \tag{5}$$

$$\frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)} = \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c}, \tag{6}$$

$$\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\cot\frac{1}{2}\gamma}, \tag{7}$$

$$\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)} = \frac{\tan\frac{1}{2}(\alpha+\beta)}{\cot\frac{1}{2}\gamma}. \tag{8}$$

Corollary 1.3 (**Law of tangents** **for spherical triangles).**

$$\frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}(a+b)} = \frac{\tan \frac{1}{2}(\alpha-\beta)}{\tan \frac{1}{2}(\alpha+\beta)}. \tag{9}$$

Corollary 1.4 (**Law I of cosines** **for spherical triangles).**

$$\cos c = \cos a \, \cos b + \sin a \, \sin b \, \cos \gamma. \tag{10}$$

Corollary 1.5 (**Law II of cosines** **for spherical triangles).**

$$\cos \gamma = -\cos \alpha \, \cos \beta + \sin \alpha \, \sin \beta \, \cos c. \tag{11}$$

Corollary 1.6 (**Law of sines** **for spherical triangles).**

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}. \tag{12}$$

Hyperbolic triangles. Consider a triangle in the hyperbolic plane H^2 having side-lengths $a, b, c > 0$ and corresponding opposite interior angles $\alpha, \beta, \gamma \in (0, \pi)$.

Theorem 1.7 (**Delambre-Gauss formulas** **for hyperbolic triangles).**

$$\cosh \tfrac{1}{2}(a+b) \, \sin \tfrac{1}{2}\gamma = \cos \tfrac{1}{2}(\alpha+\beta) \, \cosh \tfrac{1}{2}c, \tag{13}$$

$$\sinh \tfrac{1}{2}(a+b) \, \sin \tfrac{1}{2}\gamma = \cos \tfrac{1}{2}(\alpha-\beta) \, \sinh \tfrac{1}{2}c, \tag{14}$$

$$\cosh \tfrac{1}{2}(a-b) \, \cos \tfrac{1}{2}\gamma = \sin \tfrac{1}{2}(\alpha+\beta) \, \cosh \tfrac{1}{2}c, \tag{15}$$

$$\sinh \tfrac{1}{2}(a-b) \, \cos \tfrac{1}{2}\gamma = \sin \tfrac{1}{2}(\alpha-\beta) \, \sinh \tfrac{1}{2}c. \tag{16}$$

Remark. Note that $a > b$ if and only if $\alpha > \beta$.

Corollary 1.8 (**Law I of cosines** **for hyperbolic triangles).**

$$\cosh c = \cosh a \, \cosh b - \sinh a \, \sinh b \, \cos \gamma. \tag{17}$$

Corollary 1.9 (**Law II of cosines** **for hyperbolic triangles).**

$$\cos \gamma = -\cos \alpha \, \cos \beta + \sin \alpha \, \sin \beta \, \cosh c. \tag{18}$$

Corollary 1.10 (**Law of sines** **for hyperbolic triangles).**

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}. \tag{19}$$

Convex right-angled hexagons in H^2. Consider a convex right-angled hexagon in H^2 having side-lengths $l_1, \cdots, l_6 > 0$ in cyclic order.

Theorem 1.11 (**Delambre-Gauss formulas** for convex r.a.h.'s in H^2).

$$\cosh \tfrac{1}{2}(l_1 + l_3) \sinh \tfrac{1}{2} l_2 = \cosh \tfrac{1}{2}(l_4 + l_6) \cosh \tfrac{1}{2} l_5, \quad (20)$$
$$\sinh \tfrac{1}{2}(l_1 + l_3) \sinh \tfrac{1}{2} l_2 = \cosh \tfrac{1}{2}(l_4 - l_6) \sinh \tfrac{1}{2} l_5, \quad (21)$$
$$\cosh \tfrac{1}{2}(l_1 - l_3) \cosh \tfrac{1}{2} l_2 = \sinh \tfrac{1}{2}(l_4 + l_6) \cosh \tfrac{1}{2} l_5, \quad (22)$$
$$\sinh \tfrac{1}{2}(l_1 - l_3) \cosh \tfrac{1}{2} l_2 = \sinh \tfrac{1}{2}(l_4 - l_6) \sinh \tfrac{1}{2} l_5. \quad (23)$$

Remark. Note that $l_1 < l_3$ if and only if $l_4 < l_6$.

Corollary 1.12 (**Law of cosines** for convex r.a.h.'s in H^2).

$$\cosh l_n = -\cosh l_{n+2} \cosh l_{n+4} + \sinh l_{n+2} \sinh l_{n+4} \cosh l_{n+3}. \quad (24)$$

Corollary 1.13 (**Law of sines** for convex r.a.h.'s in H^2).

$$\frac{\sinh l_1}{\sinh l_4} = \frac{\sinh l_3}{\sinh l_6} = \frac{\sinh l_5}{\sinh l_2}. \quad (25)$$

2. Trigonometric formulas for right-angled hexagons in H^3

Hyperbolic 3-space: H^3.

Right-angled hexagon in H^3. A **r.a.h.** in H^3 is a cyclically indexed six-tuple $(L_1, \cdots, L_6)$ of lines in H^4 such that, for each n modulo 6, lines L_n and L_{n+1} intersect perpendicularly. It is said to be **oriented** if all the lines are oriented.

Complex (full) side-lengths σ_n of an oriented r.a.h. in H^3.
For an oriented right-angled hexagon $(\vec{L}_1, \cdots, \vec{L}_6)$ in H^3, let $\sigma_1, \cdots, \sigma_6 \in \mathbb{C}/2\pi i\mathbb{Z}$ be respectively the complex (full) side-lengths of its side-lines $\vec{L}_1, \cdots, \vec{L}_6$.

Theorem 2.1 (**Laws of cosines** for oriented r.a.h.'s in H^3).

$$\cosh \sigma_n = \cosh \sigma_{n+2} \cosh \sigma_{n+4} + \sinh \sigma_{n+2} \sinh \sigma_{n+4} \cosh \sigma_{n+3}. \quad (26)$$

Theorem 2.2 (**Laws of sines** for oriented r.a.h.'s in H^3).

$$\frac{\sinh \sigma_1}{\sinh \sigma_4} = \frac{\sinh \sigma_3}{\sinh \sigma_6} = \frac{\sinh \sigma_5}{\sinh \sigma_2}. \quad (27)$$

Remark. The above two laws for oriented r.a.h.'s in H^3 were known to **Schilling** as early as in 1891, but a correct treatment of signs seems to be given first by **Fenchel** in "Elementary Geometry in Hyperbolic Space" published in 1989.

Complex half side-lengths δ_n of an oriented r.a.h. in H^3.
For an oriented **r.a.h.** $(\vec{L}_1, \cdots, \vec{L}_6)$ in H^3, let $\delta_n \in \mathbb{C}/2\pi i\mathbb{Z}$ be an arbitrary choice of one its **two** complex half side-lengths for $\vec{L}_n$, the other being $\delta_n + \pi i \in \mathbb{C}/2\pi i\mathbb{Z}$.

We obtain Delambre-Gauss formulas for oriented right-angled hexagons in H^3.

Theorem 2.3 (**Delambre-Gauss formulas for oriented r.a.h.'s in H^3**). For an oriented **r.a.h.**in H^3, there exists $\varepsilon \in \{-1, 1\}$, depending on the choices of the half side-lengths $\delta_1, \cdots, \delta_6$, so that the following formulas (28)–(31) hold:

$$\cosh(\delta_1+\delta_3)\cosh\delta_2 = \varepsilon\cosh(\delta_4+\delta_6)\cosh\delta_5, \tag{28}$$
$$-\sinh(\delta_1+\delta_3)\cosh\delta_2 = \varepsilon\cosh(\delta_4-\delta_6)\sinh\delta_5, \tag{29}$$
$$-\cosh(\delta_1-\delta_3)\sinh\delta_2 = \varepsilon\sinh(\delta_4+\delta_6)\cosh\delta_5, \tag{30}$$
$$\sinh(\delta_1-\delta_3)\sinh\delta_2 = \varepsilon\sinh(\delta_4-\delta_6)\sinh\delta_5, \tag{31}$$

Remark. By suitably changing orientations of some of the side-lines, one may obtain the three identities (29)–(31) from the single identity (28).

3. Generalized Delambre-Gauss formulas for oriented, augmented right-angled hexagons in H^4

Hyperbolic 4-space: H^4.

Clifford algebra or **the algebra of $\{e_1, e_2\}$-quaternions**

$$\mathbb{A}_2 := \mathsf{Cl}_{0,2} = \mathbb{R} + \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_1e_2$$

subject to $e_1^2 = e_2^2 = -1$ and $e_1e_2 + e_2e_1 = 0$.

Reverse involution $()^* : \mathbb{A}_2 \to \mathbb{A}_2$ is defined by

$$(x_0 + x_1e_1 + x_2e_2 + x_{12}e_1e_2)^* := x_0 + x_1e_1 + x_2e_2 - x_{12}e_1e_2,$$

with real coefficients x_0, x_1, x_2, x_{12}.

Hyperbolic functions cosh and sinh with an $\mathbb{A}_2$-variable are defined by:

$$\cosh x := \frac{\exp(x)+\exp(-x^*)}{2}, \quad \sinh x := \frac{\exp(x)-\exp(-x^*)}{2}.$$

Line and plane in H^4. By **line** and **plane** in H^4 we mean respectively complete geodesic line and totally geodesic plane in H^4.

Right-angled hexagon in H^4. A **r.a.h.** in H^4 is a cyclically indexed six-tuple $(L_1, \cdots, L_6)$ of lines in H^4 such that, for each n modulo 6, lines L_n and L_{n+1} intersect perpendicularly. It is said to be **oriented** if all lines are oriented.

Line-plane flag. A **line-plane flag** in H^4 is an ordered pair $F=(L,\Pi)$, where L is a line and Π is a plane in H^4 such that line L is contained in plane Π. It is said to be **oriented** if both the line L and the plane Π are oriented.

We say that a line L' and a line-plane flag $F=(L,\Pi)$ intersect perpendicularly if L' intersects each of L and Π perpendicularly.

Augmented right-angled hexagon in H^4. An **a.r.a.h.** in H^4 is a cyclically indexed six-tuple $(S_1,\cdots,S_6)$ such that either S_1,S_3,S_5 are all lines and S_2,S_4,S_6 are all line-plane flags in H^4, or S_1,S_3,S_5 are all line-plane flags and S_2,S_4,S_6 are all lines in H^4, and such that, for each n modulo 6, S_n and S_{n+1} intersect perpendicularly. It is said to be **oriented** if all S_n, $n=1,\cdots,6$ are oriented.

Two e_2-complex half distances $\delta_{\vec{F}}(\vec{L}_1,\vec{L}_2)\in(\mathbb{R}+\mathbb{R}e_2)/2\pi e_2\mathbb{Z}$ from $\vec{L}_1$ to $\vec{L}_2$ along a common perpendicular $\vec{F}=(\vec{L},\vec{\Pi})$ in H^4.

The two values of $\delta_{\vec{F}}(\vec{L}_1,\vec{L}_2)$ differ by πe_2.

Two $\{e_1,e_2\}$-quaternion half distances $\delta_{\vec{L}}(\vec{F}_1,\vec{F}_2)\in\mathbb{A}_2 \bmod (\text{period})$ from $\vec{F}_1$ to $\vec{F}_2$ along a common perpendicular $\vec{L}$ in H^4.

The two values of $\delta_{\vec{L}}(\vec{F}_1,\vec{F}_2)$ differ by πu for some $u\in\sqrt{-1}\subset\mathbb{A}_2$.

Theorem 3.1 (**Delambre-Gauss formulas for oriented a.r.a.h.'s in H^4**)**.** For an oriented, augmented right-angled hexagon $(\vec{L}_1,\vec{F}_2,\vec{L}_3,\vec{F}_4,\vec{L}_5,\vec{F}_6)$ in H^4 with arbitrary choices of $\{e_1,e_2\}$-quaternion half side-lengths $\delta_1,\delta_3,\delta_5$ and arbitrary choices of e_2-complex half side-lengths $\delta_2,\delta_4,\delta_6$, the following formulas hold:

$$\begin{aligned}
&(\sinh\delta_1\cosh\delta_2\sinh\delta_3+\cosh\delta_1\cosh\delta_2\cosh\delta_3)^*\\
&\qquad= \varepsilon\,(\sinh\delta_4\cosh\delta_5\sinh\delta_6+\cosh\delta_4\cosh\delta_5\cosh\delta_6); &(32)\\
&(\sinh\delta_1\sinh\delta_2\sinh\delta_3-\cosh\delta_1\sinh\delta_2\cosh\delta_3)^*\\
&\qquad= \varepsilon\,(\sinh\delta_4\cosh\delta_5\cosh\delta_6+\cosh\delta_4\cosh\delta_5\sinh\delta_6); &(33)\\
&(\sinh\delta_1\cosh\delta_2\cosh\delta_3+\cosh\delta_1\cosh\delta_2\sinh\delta_3)^*\\
&\qquad= \varepsilon\,(\sinh\delta_4\sinh\delta_5\sinh\delta_6-\cosh\delta_4\sinh\delta_5\cosh\delta_6); &(34)\\
&(\sinh\delta_1\sinh\delta_2\cosh\delta_3-\cosh\delta_1\sinh\delta_2\sinh\delta_3)^*\\
&\qquad= \varepsilon\,(\sinh\delta_4\sinh\delta_5\cosh\delta_6-\cosh\delta_4\sinh\delta_5\sinh\delta_6), &(35)
\end{aligned}$$

with $\varepsilon=1$ or -1, depending on the choices of the six half side-lengths $\{\delta_n\}_{n=1}^6$.

Remark. Formulas (32)–(35) above can be abbreviated as follows:

$$\begin{aligned}
(\mathbf{scs}+\mathbf{ccc})^*_{123} &= \varepsilon\,(\mathbf{scs}+\mathbf{ccc})_{456};\\
(\mathbf{sss}-\mathbf{csc})^*_{123} &= \varepsilon\,(\mathbf{scc}+\mathbf{ccs})_{456};\\
(\mathbf{scc}+\mathbf{ccs})^*_{123} &= \varepsilon\,(\mathbf{sss}-\mathbf{csc})_{456};\\
(\mathbf{ssc}-\mathbf{css})^*_{123} &= \varepsilon\,(\mathbf{ssc}-\mathbf{css})_{456}.
\end{aligned}$$

Remark. The formulas (32)–(35) above are left invariant under taking the reverse involution $()^*$ and shifting the indices by $123456\to456123$.

4. Ideas of Proof

Theorem 4.1. For an oriented **r.a.h.** $(\vec{L}_1, \vec{L}_2, \cdots, \vec{L}_6)$ in H^3, let $\mathbf{M}_n \in \mathrm{Isom}^+(\mathsf{H}^3)$, n modulo 6, be such that $\mathbf{M}_n(\vec{L}_n) = \vec{L}_n$ and $\mathbf{M}_n(\vec{L}_{n-1}) = \vec{L}_{n+1}$. Then

$$\mathbf{M}_6\,\mathbf{M}_5\,\mathbf{M}_4\,\mathbf{M}_3\,\mathbf{M}_2\,\mathbf{M}_1 \;=\; \mathrm{Id}. \tag{36}$$

Theorem 4.2. For an oriented **r.a.h.** $(\vec{L}_1, \vec{L}_2, \cdots, \vec{L}_6)$ in H^3, let $\mathbf{M}_n \in \mathrm{Isom}^+(\mathsf{H}^3)$, n modulo 6, be as in Theorem 4.1 and let $\mathbf{T}_n \in \mathrm{Isom}^+(\mathsf{H}^3)$ be a conjugate of $\mathbf{M}_n$ such that $\mathbf{T}_n(\vec{L}_1) = \vec{L}_1$ if $n = 1, 3, 5$ and $\mathbf{T}_n(\vec{L}_2) = \vec{L}_2$ if $n = 2, 4, 6$. Then

$$\mathbf{T}_1\,\mathbf{T}_2\,\mathbf{T}_3\,\mathbf{T}_4\,\mathbf{T}_5\,\mathbf{T}_6 \;=\; \mathrm{Id}. \tag{37}$$

Theorem 4.3. For an oriented **a.r.a.h.** $(\vec{S}_1, \cdots, \vec{S}_6)$ in H^4, let $\mathbf{M}_n \in \mathrm{Isom}^+(\mathsf{H}^4)$, n modulo 6, be such that $\mathbf{M}_n(\vec{S}_n) = \vec{S}_n$ and $\mathbf{M}_n(\vec{S}_{n-1}) = \vec{S}_{n+1}$. Then

$$\mathbf{M}_6\,\mathbf{M}_5\,\mathbf{M}_4\,\mathbf{M}_3\,\mathbf{M}_2\,\mathbf{M}_1 \;=\; \mathrm{Id}. \tag{38}$$

Theorem 4.4. For an oriented **a.r.a.h.** $(\vec{S}_1, \cdots, \vec{S}_6)$ in H^4, let $\mathbf{M}_n \in \mathrm{Isom}^+(\mathsf{H}^4)$, n modulo 6, be as in Theorem 4.3 and let $\mathbf{T}_n \in \mathrm{Isom}^+(\mathsf{H}^4)$ be a conjugate of $\mathbf{M}_n$ such that $\mathbf{T}_n(\vec{S}_1) = \vec{S}_1$ if $n = 1, 3, 5$ and $\mathbf{T}_n(\vec{S}_2) = \vec{S}_2$ if $n = 2, 4, 6$. Then

$$\mathbf{T}_1\,\mathbf{T}_2\,\mathbf{T}_3\,\mathbf{T}_4\,\mathbf{T}_5\,\mathbf{T}_6 \;=\; \mathrm{Id}. \tag{39}$$

Proof of Delambre-Gauss formulas for oriented a.r.a.h.'s in H^4. In the upper half-space model of $\mathsf{H}^{n+2} \equiv \mathbb{R} + \mathbb{R}e_1 + \cdots + \mathbb{R}e_n + \mathbb{R}^+ e_{n+1}$, we have

$$\mathrm{Isom}^+(\mathsf{H}^n) \;\equiv\; \mathrm{PSL}(2, \Gamma_n \cup 0),$$

where $\Gamma_n \subset \mathbb{A}_n^\times$ is the full Clifford group and a Vahlen matrix $A \in \mathrm{SL}(2, \Gamma_n \cup 0)$ acts on H^{n+2} as a fractional linear transformation:

$$Ax = (ax + b)(cx + d)^{-1}.$$

Note that $\Gamma_1 \cup 0 = \mathbb{A}_1 \equiv \mathbb{C}$ and $\Gamma_2 \cup 0 = \mathbb{A}_2$. Now choose special positions for $\vec{S}_1$ and $\vec{S}_2$ as follows:

$$\vec{S}_1 = \vec{L}_1 = \vec{L}_{[0,\infty]}; \quad \vec{S}_2 = \vec{F}_2 = (\vec{L}_{[-1,1]}, \vec{\Pi}_{[-1,1]\vee[-e_1,e_1]}).$$

We obtain an identity of 2×2 matrices by replacing each isometry T_n in (39) by a Vahlen matrix A_n, and the identity isometry by εI for some $\varepsilon \in \{-1, 1\}$. Precisely, we have

$$A_1 A_2 A_3 \;=\; \varepsilon\,(A_4 A_5 A_6)^{-1}.$$

Working out the product matrices on both sides and equating the corresponding entries, we obtain the Delambre-Gauss formulas by suitable manipulations. □

THANK YOU!