Lecture 2 The wild solutions of DeLellis and Szekelyhidi

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Introduction

The purpose of this lecture is the description of the construction of very singular solutions (in any space dimension n) of the incompressible Euler equation.

Theorem Let $\Omega \subset\subset \mathbb{R}^n$, $0 < T < \infty$ and $(x,t) \mapsto \overline{e}(x,t) > 0$ a continuous function with support in $\overline{\Omega \times]0,T[}$ then for any $\eta > 0$ there exists a weak solution (u,p) of the Euler equation with the following properties

- $u \in C(\mathbb{R}_t; L^2_w(\mathbb{R}^n))$;
- $\frac{|u(x,t)|^2}{2} = -\frac{n}{2} p(x,t) = e(x,t)$
- $\sup_{t\in\mathbb{R}} \|u(.,t)\|_{H^{-1}(\mathbb{R}^n)} \leq \eta$
- $(u,p) = \lim_{k\to\infty}(u_k,p_k)$ in $L^2(dx,dt)$ with $(u_k,p_k) \in C^\infty$ compact support solution of the Euler equation with a convenient forcing f_k converging to 0 in $\mathcal{D}'$.

Comments on the DeLellis -Szekelyhidi Constructions

- On one hand the above theorem shows how non physical is the incompressible Euler Equation. It generate solutions starting from nothing dying after a finite time and in the mean time having their own energy thus solving the energy crisis....
- On the other since the Euler equation is the "limit in many senses " of more classical equations (incompressible and compressible Navier-Stokes equations, Boltzmann equation and so on...) this shows how unstable such more realistic formulation may become when some scaling parameters go to zero.
- This theorem had several forerunners more precise due to Sheffer and Shnirelman... However all these constructions share in common the use of accumulation of terms with small amplitude and large frequencies.

Comments on the DeLellis

Both the statement and some steps of the proof share common point with the problem of isometric imbedding:

- Nash-Kuiper: For any $n \in N$ and $r \in]0,1[$ there exists an isometric imbedding C^1 from $S^n(1)$ in $B^{n+1}(r)$
- Cohn-Vossen: The above statement is not true if C^2 regularity is required!!
- Therefore in both problem appear an issue of threshold of regularity.
- For the isometric imbedding the exact threshold is not fully determined.
- For regular solutions of the Euler equation C^0 is a threshold in the class of Holder and Besov spaces...
- For weak solution $\mathcal{B}^{\frac{1}{3}}_{3,co}$ seems to be a threshold for conservation of energy (at least any solution with this regularity conserves the energy)...
- The construction provides, with corollary, solutions that will both violate conservation of energy and uniqueness of Cauchy problem.

Main steps of the proof

- Differential inclusion
- Plane wave solutions with Tartar wave cone
- Λ convex hull of the wave cone
- "Localised plane waves"
- Subsolutions and functionnals
- Improvement of the functionnals
- Completion of the proof

h-Principle

The proof consists in decoupling linear evolution and non linear constraint by the introduction of a linear system and $u \in \mathcal{L}$ and a constraint $\mathcal{K} = \{u$ such that $\mathcal{F}(u) = 0\}$. The sub solutions $u \in \mathcal{K}^c$ (the convex hull or as it will be shown below the Λ convex hull of $\mathcal{K}$) are the functions $u \in \mathcal{L}$ such that $F(u) \leq 0$. Then there will be two methods.

1 Starting from an element $u_0 \in \mathcal{L} \cup \mathcal{K}^c$ contruct a sequence $u_k \in \mathcal{K}^c$ such that $\mathcal{F}(u_k) < 0\,, \lim_{k\to\infty} \mathcal{F}(u_k) = 0$.

2 Define on $\mathcal{K}^c \cap \mathcal{L}$ a convenient metric topology for which the function F is lower semi continous. Hence its points of continuity form a residual Baire set. Then one shows that the points of continuity must satisfy the relation $F(u) = 0$. In both case one shows that $\mathcal{K} \subset \mathcal{K}^c$ is "big" enough. For that one uses special oscillatory solutions (plane waves, contact discontinuities) which are closely related to the constructions of the forerunners. In the proofs below, for convenience $\mathcal{F}$ is changed into $J = -\mathcal{F}$ implying the change of lower semi continuty into upper semi continuity and so on..

Differential inclusion

I_n the $n \times n$ identity matrix and $\mathcal{S}_0^n$ the space of real valued symmetric matrices with 0 Starting point is the following evident proposition:

Proposition 1 The two following systems are equivalent:

$$(v, p) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}^n \times \mathbb{R}^n)$$
$$\partial_t v + \nabla \cdot (v \otimes v) + \nabla p = 0\,, \nabla \cdot v = 0$$

$$(v, M, q) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R})$$
$$\partial_t v + \nabla \cdot M + \nabla q = 0\,, \nabla q = 0\,, \nabla \cdot v = 0\ , q = p + \frac{|v|^2}{n}$$
$$M = v \otimes v - \frac{|v|^2}{n} I_n \text{ almost every where}$$

System uncoupled a first order pde and a constraint described by

$$K = \left\{ (v, M) \in \mathbb{R}^n \times \mathcal{S}_0^n\, ; M = v \otimes v - \frac{|v|^2}{n} I_n \right\}\ K_r = \left\{ (v, M) \in K\, ; |v| = r \right\}$$

Plane wave solutions with Tartar wave cone

Tartar wave cone $\Lambda = \{(v_0, M_0, q_0) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}\} \Leftrightarrow \exists (v, M, q)(x,t) = (v_0, M_0, q_0) h(\xi \cdot x + ct)$ solution of the linear problem:
Proposition 2

- $$\Lambda = \left\{ (v, M, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R};\ \det \begin{bmatrix} M + qI_n & v \\ v & 0 \end{bmatrix} = 0 \right\}$$
- $\forall (v, M) \in \mathbb{R}^n \times \mathcal{S}_0^n \ \ \exists q$ such that $(v, M, q) \in \Lambda$;
- $\forall v_0 \in \mathbb{R}^n \ \ \exists p_0, \xi$ such that $(v_0, p_0) h(\dfrac{x \cdot \xi}{\epsilon})$ stationnary plane wave sol.

Comments

- The wave cone is very big it contains solutions (even time independent) with spatial oscillations collinear to any direction.
- Below are considered special plane waves associated to rang 2 matrices. They are time dependent but with prescribed velocity and pressure:

$$\frac{|v(x,t|^2}{2} = -\frac{n}{2}p(x,t) = e(x,t) \quad \text{a priori prescribed}$$

- They will generate the convex hull of K .

Basic plane waves

$$a, b \in \mathbb{R}^n\,,\ a \neq b|a| = |b|\ \Rightarrow (a-b, a\otimes a - b\otimes b, 0) \in \Lambda$$

Proof

$$z \in (a-b)^{\perp}, c = z\cdot a = z\cdot b \Rightarrow \ \det \begin{bmatrix} a\otimes a - b\otimes b & a-b \\ a-b & 0 \end{bmatrix} \begin{bmatrix} z \\ c \end{bmatrix} = 0$$

$$\Lambda_r = \left\{ tW(a,b); |a| = |b| = r\ ;\ a \neq \pm b, t \geq 0 \right\}$$

Λ convex hull of K

K' Λ convex hull of K smallest set $K' \supset K$ such that

$$\forall a, b \in K', b - a \in \Lambda \Rightarrow [a, b] \subset K'$$

Proposition 3

(i) For any $r > 0$ the Λ convex hull of K_r coincides with the convex hull of K_r which is equal to

$$K_r^{co} = \left\{ (v, M) \in \mathbb{R}^n \times \mathcal{S}_0^n : \quad |v| \leq r, (v \otimes v - M) \leq \frac{r^2}{n} I_n \right\} \quad (1)$$

$$\text{and } K_r = K_r^{co} \cap \{|v| = r\} \quad (2)$$

(ii) There is a constant $C = C(n) > 0$ such that for any $r > 0$ and $z = (v, M)$ in the interior of K_r^{co} there exists $\lambda = (\overline{v}, \overline{M}) \in \Lambda_r$ such that

$$[z - \lambda, z + \lambda] \subset \text{int} K_r^{co}$$

$$|\overline{v}| \geq \frac{C}{r}(r^2 - |v|^2) \text{ and } \text{dist}\,([z - \lambda, z + \lambda], \partial K_r^{co}) \geq \frac{1}{2}\,\text{dist}\,(z, \partial K_r^{co})$$

Comments

- K_r^{co} is for the Euler equation a set of *subsolution*
- In particular $0 \in K_r^{co}$ is a subsolution. Therefore wild solutions will be constructed from 0.
- The point (ii) says that as long as a subsolution is not on the boundary (a solution) it is the center of a segment of size bounded from below and this will be used to add oscillations to make it converge to the boundary.

Proof of (i)

Let C_r the right hand side of (1) One has

$$K_r \subset C_r$$

then one shows (a) C_r is convexe; (b) C_r is compact; (c) K_r contains all the extremal points of C_r then the Krein-Rutman theorem implies $K_r^{co} = C_r$

$$(v, M) \mapsto \Phi(v, M) = \sigma_{max}(v \otimes v - M) = \max_{\xi \in S^{n-1}(1)} ((\xi \cdot v)^2 - (M\xi, \xi))$$

$$\Phi(v, M) \text{convex and } C_r = \Phi^{-1}([0, \frac{r^2}{n}]) \cap \{|v| \leq r\} \Rightarrow \text{ convexity } \Rightarrow (a)$$

$$M \geq v \otimes v - \frac{r^2}{n} I_n \geq -\frac{r^2}{n} I_n \text{ trace}(M) = 0 \Rightarrow \text{Compacity}$$

For (c) write $v \otimes v - M = \mathrm{diag}(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)$ and show that any point with $|v| < r$ and $\lambda_n < r^2/n$ is not extremal.

Localised Plane waves

For any $r > 0$ and $\lambda = W(a,b) \in \Lambda_r, |a| = |b| = r > 0, b \neq \pm a$ introduce the time dependent 3 order differential operator
$A(\nabla) = (A_v(\nabla), A_M(\nabla)) : C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R} \times \mathcal{S}_0^n)$:

$$A_v^i(\nabla) = \sum_{k,l} (a^i b^k - b^i a^k)\partial_{kll}$$

$$A_M^{ij}(\Delta) = \sum_k (b^i a^k - a^i b^k)\partial_{tkj} + \sum_k (b^j a^k - a^j b^k)\partial_{tki}$$

Proposition 4
(i) For any $\phi \in C_c^\infty(\mathbb{R}^{n+1})$ $A(\nabla)(\phi)$ is a solution of the linear system:

$$\nabla \cdot A_v(\nabla)(\phi) = 0\,, \partial_t A_v(\nabla)(\phi) + \nabla \cdot A_M(\nabla)(\phi) = 0$$

(ii) With $\phi(x,t) = \psi(\frac{(a+b)\cdot x - st}{\epsilon})$ with $s = \frac{|a+b|^2}{2} = r^2 + a \cdot b$

$$A(\nabla)(\phi) = 2s^2\epsilon^{-3}((a-b), (a\otimes a - b \otimes b))\psi'''(\frac{(a+b)\cdot x - st}{\epsilon})$$

Corollary

For any $r > 0$, $\lambda \in \Lambda_r$ and any $\psi \in C_c^\infty(\mathbb{R})$ there exists $(\xi, c) \in \mathbb{R}^n \times \mathbb{R}$, $\xi \neq 0$ such that

$$\text{with } \phi(x,t) = \psi(\xi \cdot x + ct)\,, A(\nabla)\phi = \lambda\psi(\xi \cdot x + ct)$$

Proof: Above take: In the above formula take:

$$\epsilon = \Big(\frac{|a+b|^4}{2}\Big)^{\frac{1}{3}}, \quad \xi = \frac{a+b}{\epsilon}, \quad c = -\frac{|a+b|^2}{2\epsilon}\,.$$

Localised plane waves, Proposition 4

Let $\mathcal{O} \subset \mathbb{R}^n$ open bounded subset of $\mathbb{R}^n$,
$I =]t_0, t_1[\subset \mathbb{R}, r > 0, \lambda = (\overline{v}, \overline{M}) \subset \Lambda_r \quad \mathcal{V} \subset \mathbb{R}^n \times \mathcal{S}^n_0$.
Let $\mathcal{O}' \subset\subset \mathcal{O}, \theta \in [0, (t_1 - t_0)/2], I_\theta = [t_0 + \theta, t_1 - \theta]$.
Then for any $\eta > 0$ there exists

$$(v, M, 0) \in C^\infty_c(\mathcal{O} \times I; \mathcal{V})$$

solution of the linear system with:

$$\forall t\,,\ \|v(.,t)\|_{H^{-1}(\mathbb{R}^n)} \leq \eta \ \text{ and } \ \inf_{t \in I_\theta} \frac{1}{|\mathcal{O}'|} \int_{\mathcal{O}'} |v(x,t)|^2 dx \geq \frac{|\overline{v}|^2}{3}$$

Proof

Introduce $\phi(x,t)$ with compact support in $\mathcal{O} \times I$ equal to 1 in $\mathcal{O}' \times I_\theta$. With $\lambda = (\overline{v}, \overline{M})$ introduce ξ, c as above.

$$z_\epsilon(x,t) = (v_\epsilon, M_\epsilon,)(x,t) = A(\nabla)[\epsilon^3 \phi(x,t) \cos(\frac{\xi \cdot x + ct}{\epsilon})]$$

$$\text{Leibnitz formula} \ \Rightarrow z_\epsilon(x,t) = \lambda \sin(\frac{\xi \cdot x + ct}{\epsilon}) + O(\epsilon)$$

On $\mathcal{O}'$ use

$$\frac{1}{|\mathcal{O}'|}\int_{\mathcal{O}'} |v(x,t)|^2 dx = |\overline{v}|^2 \frac{1}{|\mathcal{O}'|}\int_{\mathcal{O}'} \sin^2(\frac{\xi \cdot x + ct}{\epsilon}) dx + O(\epsilon) > \frac{|\overline{v}|^2}{3} + O(\epsilon)$$

Eventually use for $\zeta \in H^1(\mathbb{R}^n)$

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} z_\epsilon(x,t)\zeta(x) dx = 0$$

Space of super-solutions

$$X_0 = \{z = (v, M) \in C_c^\infty(\Omega \times]0, T[; \mathbb{R}^n \times S_0^n)\}$$
$$\partial_t v + \nabla \cdot M = 0\,, \nabla \cdot v = 0 \;\forall (x,t) z(x,t) \in \text{int } K^{co}_{\sqrt{2e(x,t)}}$$
$$\forall (\Omega_0 \subset\subset \Omega, \tau \in]0, T/2[)\; J_{\tau,\Omega_0} = \sup_{\tau \le t \le T-\tau} \int_{\Omega_0} [e(x,t) - \frac{|v(x,t)|^2}{2}] dx$$

Proposition 5

$(i) z = (v, M) \in X_0$ and $p = -\frac{|v|^2}{n} \Rightarrow (v, p)$ solution of the Euler equation with a forcing term $f = \nabla \cdot (v \otimes v - \frac{|v|^2}{n} - M) \in C_c^\infty(\Omega \times]0, T[; \mathbb{R}^n)$.
$(ii) z_k = (v_k, M_k)_{k \in \mathbf{N}} \to z = (v, M)$ a sequence of elements of X_0 converging in $C(]0, T[; L^2_{\mathrm{loc}}(\Omega))$ such that:

$$\forall (\tau, \Omega_0)\,,\; J_{\tau,\Omega_0} \to 0\,.$$

Then $v \in C(\mathbb{R}; L^2_w(\mathbb{R}^n))$ is a weak solution of the Euler equation which satisfies $\frac{|v(x,t)|^2}{2} = e(x,t) = -\frac{n}{2} p(x,t)$ and which in particular is 0, outside $\overline{\Omega} \times [0, T]$.

Proof

The fact that $v \in C(\mathbb{R}; L^2_n(\mathbb{R}^n))$ is a consequence of Proposition 3 and the fact that it is a solution is a consequence of Proposition 5.
The construction of the sequence involves two steps...
First a step of improvement and second a step of iteration

Improvement Proposition 6

A finite sequence $1 \leq l \leq L$ of increasing open sets $\Omega_l \times]\tau_l, T - \tau_l[$ with: $0 < \tau_L < \ldots < \tau_l < \ldots < \tau_1 \, , \overline{\Omega}_1 \subset\subset \Omega_l \subset\subset \Omega_L$ Assume that

$$\forall l \ , J_{\tau_l, \Omega_l}(v) > 0 \, . \qquad (3)$$

Then for every $\eta > 0$ there exists an element $z' = (v', M')$ such that:

$$\|z' - z\|_{C([0,T];H^{-1}(\Omega))} \leq \eta \qquad (4)$$

$$\forall 1 \ \leq l \leq L, J_{\tau_l, \Omega_l}(v') \leq J_{\tau_l, \Omega_l}(v) - \beta(J_{\tau_l, \Omega_l}(v)) \qquad (5)$$

with in (5) $\beta(\alpha)$ denoting a positive increasing function which with α small behaves like $C\alpha^2$

Iteration

In the spirit Nash-Moser theorem: A sequence of regularizing function $\rho_{\epsilon_j}(x,t)$ Assume for $j \le k-1$ $z_j = (v_j, M_j), \epsilon_j$ such that

$$J_{\tau_j,\Omega_j}(v_{k-1}) \le J_{\tau_j,\Omega_j}(v_{k-2}) - \beta_j(J_{\tau_j,\Omega_j}(v_{k-2}))\,\forall j \le k-2$$

$$\sup_t \|(z_l - z_{l-1}) \star \rho_{\epsilon_j}\|_{L^2(\Omega)} < 2^{-l} \forall j \le l \le k-1$$

Then with the proposition 6 choose z_k such that

$$J_{\tau_j,\Omega_j}(v_k) \le J_{\tau_j,\Omega_j}(v_{k-1}) - \beta_j(J_{\tau_j,\Omega_j}(v_{k-1}))\,\forall j \le k$$

$$\sup_t \|(z_k - z_{k-1})\|_{H^{-1}(\Omega)} \le \eta_k$$

with η_k small enough to imply

$$\sup_t \|(z_k - z_{k-1}) \star \rho_{\epsilon_j}\|_{L^2(\Omega)} < 2^{-(k-1)} \forall j \le k-1\,;\, \sup_t \|(z_k - z_{k-1})\|_{H^{-1}(\Omega)} \le \eta 2^{-k}$$

Eventually choose ϵ_k such that

$$\|z_j - z_j \star \rho_{\epsilon_k}\| < 2^{-k} \forall j \le k$$

Iteration End of Proof.

The sequence (z_k) is bounded in $L^2(\mathbb{R}^n \times \mathbb{R})$ hence converges weakly to $z \in L^2(\mathbb{R}^n \times \mathbb{R})$ Moreover $\sup_t \|z\|_{H^{-1}(\Omega)} \leq \eta$ For (τ_j, Ω_j) and $k \geq j$ one has in $C(\tau_j, T - \tau_j]; L^2(\Omega_j))$

$$\|z_k - z\| \leq \|z_k - z_k \star \rho_{\epsilon_k}\| + \|z_k \star \rho_{\epsilon_k} - z \star \rho_{\epsilon_k}\| + \|z \star \rho_{\epsilon_k} - z\| \qquad (6)$$

Hence with the iteration process strong convergence in

$$C([\tau_j, T - \tau_j]; L^2(\Omega_j))$$

Now if $l_j = \overline{\lim}_{k\to\infty} J_{\tau_j,\Omega_j} > 0$ with the improvement process one has

$$l_j \leq l_j - \beta_j(l_j)$$

hence a contradiction.

One step improvement

Proposition

$$\Omega_0 \times]\tau, T-\tau[\subset\subset \Omega \times]0, T[\ , z = (v, M) \in X_0$$

with

$$J_{\tau,\Omega_0}(v) \geq \alpha > 0 \ \ \alpha \in]0, 1[$$

Then for any $\eta > 0$ there exists an element $z' = (v', M') \in X_0$ such that

$$\|z' - z\|_{C([0,T];H^{-1}(\Omega))} \leq \eta \text{ and } J_{\tau,\Omega_0}(v') \leq J_{\tau,\Omega_0}(v) - \beta(\alpha)$$

Proof of the one step improvement for fixed value of t

Start with a convenient covering by $\mathcal{N}$ cubes such that on each cube the oscillation of z is bounded by $\alpha/10$ with notational abuse denote by $C' \subset\subset C \in \mathcal{C}$, $C' = 0.9C$ cubes their centers, sub cubes and introduce $c > 0$ such that

$$c \le \frac{1}{40|C'|\mathcal{N}}$$

With oscillations and Riemann sum type construction one has:

$$\sum_{e(C)-\frac{|v(C)|^2}{2} \ge c\alpha} \{|C'|(e(C) - \frac{|v(C)|^2}{2})\} \ge \frac{\alpha}{5} \tag{7}$$

$$(v', M') = (v, M) + \sum_{\mathcal{C}} (v_C, M_C)\,, \text{support}(v_C, M_C) \subset\subset C'$$

$$J_{\tau,\Omega_0}(v) - J_{\tau,\Omega_0}(v') =$$

$$\int_{\Omega_0} (e(x,t) - \frac{|v(x,t)|^2}{2})dx - \int_{\Omega_0} (e(x,t) - \frac{|v'(x,t)|^2}{2})dx$$

$$= \int_{\Omega_0} (\frac{|v'(x,t)|^2}{2} - \frac{|v(x,t)|^2}{2})dx$$

$$= \sum_{e(C) - \frac{|v(C)|^2}{2} \geq c\alpha} \int_C \frac{|v_C(x,t)|^2}{2} dx + \int_C v(x,t) \cdot v_C(x,t) dx$$

$$\geq \sum_{e(C) - \frac{|v(C)|^2}{2} \geq c\alpha} \int_C \frac{|v_C(x,t)|^2}{2} dx$$

$$- \sum_{e(C) - \frac{|v(C)|^2}{2} \geq c\alpha} \|v(.,t)\| \|v_C(.,t)\|_{H^{-1}(C)}$$

Use Proposition 3 (ii)

There exists $\lambda = (\overline{v}, \overline{M})$ such that

$$z(C) + [-\lambda, \lambda] \subset\subset K^{co}_{\sqrt{2e(x,t)}} \; |\overline{v}| \geq C \frac{(e(C) - \frac{|v(C)|^2}{2}}{\|e\|_\infty}$$

By continuity there exists a neighborhood $\mathcal{V}$ of $[-\lambda, \lambda]$ such that

$$z(x,t) + \mathcal{V} \subset \mathrm{int} K^{co}_{\sqrt{2e(x,t)}} \forall (x,t) \in C$$

With the proposition 4 one constructs a localised solution z_C with support in C value in $\mathcal{V}$ and such that

$$\sup_t \|v_C(.t)\|_{H^{-1}} \text{ small enough} \quad (8)$$

$$\frac{1}{|C'|} \int |v_C(x,t)|^2 dx \geq \frac{|v(C)|^2}{3} \geq Cte(e(C) - \frac{|v(C)|^2}{2}) \quad (9)$$

With $\sum_{e(C) - \frac{|v(C)|^2}{2} \geq c\alpha} \{|C'|(e(C) - \frac{|v(C)|^2}{2})\} \geq \frac{\alpha}{5}$ the proof is completed .