Generalizations of the Kunen Inconsistency

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The Kunen Inconsistency

Many large cardinal axioms assert the existence of a nontrivial elementary embedding $j : V \to M$.

As the axioms becomes stronger, M exhibits increasing affinity with V.

Reinhardt, taking the natural limit of this trend, proposed a nontrivial elementary embedding $j : V \to V$.

Shortly after it was proposed, Kunen refuted the existence of such embeddings j.

The Kunen inconsistency

Theorem (The Kunen Inconsistency, 1971)

There is no nontrivial elementary embedding $j : V \to V$. Consequently, there are no Reinhardt cardinals.

The theorem has been generalized by many mathematicians: Woodin, Foreman, Harada, Zapletal, Suzuki, and others.

In this talk, I shall present several known results along with some new generalizations.

The talk could have been called "generalizations-of-generalizations" of the Kunen inconsistency.

A sample of the generalizations

1. There is no $j : V[G] \to V$; nor $j : V \to V[G]$.
2. More generally, there is no j between two ground models.
3. There is no $j : M \to N$, if M, N eventually stationary correct.
4. There is no $j : V \to \mathrm{HOD}$.
5. There is no $j : V \to \mathrm{HOD}^\eta$, no $j : V \to \mathrm{gHOD}$, no $j : V \to \mathrm{gHOD}^\eta$.
6. There are no such j added by set forcing.
7. If $j : V \to M$ is elementary, then $V = \mathrm{HOD}(M)$.
8. There is no $j : \mathrm{HOD} \to V$.
9. There is no $j : M \to V$, if M is definable.
10. There is no $j : \mathrm{HOD} \to \mathrm{HOD}$ definable from parameters.

Other results: iterated HOD^η, the generic-HOD and gHOD^η, generic embeddings, definable embeddings, and $\neg\mathrm{AC}$ results.

Dispelling a few meta-mathematical clouds

Let me begin by clearing up a few meta-mathematical issues.

The first is that the Kunen inconsistency is explicitly a second-order claim

"there is no j such that..."

Since j is a proper class of some kind, this is explicitly quantifying over classes.

How are we to formalize the assertion in set theory?

A second-order claim

To be sure, many large cardinal notions have second-order definitions, with first-order equivalent formulations.

Example: measurable cardinals.

Observation

Reinhardt cardinals have no consistent first-order formulation.

Proof.

Let κ be the least Reinhardt cardinal. So there is $j : V \to V$ with critical point κ. By elementarity, $j(\kappa)$ is also the least Reinhardt cardinal, a contradiction. $\square$

Similarly, no consistent first order property can imply that κ is Reinhardt.

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Classes in ZFC

The traditional approach to classes in ZFC is via the first-order formulas that might define them. All talk of classes is a substitute for formulas.

With this approach, the Kunen inconsistency becomes a theorem scheme, asserting of each formula ψ that for no parameter z does the relation $\psi(x, y, z)$ define a function

$$y = j(x)$$

that is an elementary embedding from V to V.

Our view is that this does not convey the full power of the Kunen inconsistency.

To refute definable $j : V \to V$ is much easier than Kunen's argument, and requires neither AC nor infinite combinatorics.

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Formalize in GBC

A stronger result is obtained by formalizing the Kunen inconsistency in second-order set theory, such as Gödel-Bernays or Kelly-Morse.

Kunen himself used Kelly-Morse set theory:

> *It is intended that our results be formalized within the second order Morse-Kelley set theory..., so that statements involving the satisfaction predicate for class models can be expressed. (Kunen, 1971)*

But actually, GBC suffices, a fragment of $\mathrm{ZFC}(j)$.

A second cloud

In GBC, how do we formalize the claim that j is elementary?

Naïvely, this is a scheme:

$$\forall x\,[\varphi(\vec{x}) \longleftrightarrow \varphi(j(\vec{x}))].$$

But a scheme does not serve our purpose, since the assertion that j is elementary appears negatively in the theorem, and the negation of a scheme is not expressible as a scheme.

Kunen addressed the issue by using KM, in which first-order satisfaction is expressible.

Elementarity in GBC

In the weaker theory GBC, one can use Gaifman's observation:

Lemma (Gaifman)

If $j : M \to N$ is Δ_0-elementary and cofinal, where M and N satisfy ZF, then j is fully elementary.

Note that when the models have the same ordinals, then Σ_1-elementary embeddings are cofinal.

The conclusion of the lemma is a scheme, but the hypothesis is a first-order assertion.

Note that in KM formalization, we get full elementarity internally to the theory. For example, we can induct on Σ_n-elementarity.

In GBC, this induction takes pace in the meta-theory.

Formalizing the Kunen inconsistency

To summarize:

We formalize and prove the Kunen inconsistency in GBC as the claim that there is no class j which is a nontrivial Σ_1-elementary embedding $j : V \to V$.

Generalizations of the Kunen inconsistency

Let's begin now to prove various generalizations of the Kunen inconsistency.

Begin with the case of an elementary embedding

$$j : V[G] \to V,$$

which is a very natural case to consider. From the forcing extension $V[G]$, such an embedding maps the universe into a transitive inner model, which might seem initially like an ordinary large cardinal situation.

Embeddings $j : V[G] \to V$

Theorem (Woodin)

In any set-forcing $V[G]$, there is no $j : V[G] \to V$.

Proof.

Suppose $j : V[G] \to V$ via $\mathbb{P}$. Find $\lambda > |\mathbb{P}|$, $\kappa = \mathrm{cp}(j)$ with $j(\lambda) = \lambda$ and hence $j(\lambda^+) = \lambda^+$. In $V[G]$ partition $\mathrm{Cof}_\omega\,\lambda^+ = \bigsqcup_{\alpha<\kappa} S_\alpha$ into stationary sets. Let $S^* = j(\vec{S})(\kappa)$, a stationary subset of $(\mathrm{Cof}_\omega\,\lambda^+)^V$ in V, disjoint from every $j(S_\alpha)$. Let $C = \{\,\beta < \lambda^+ \mid j\,``\,\beta \subseteq \beta\,\}$, club subset of $\lambda^+$. Find club $D \subseteq C$ with $D \in V$. So $\exists\beta \in D \cap S^*$. Observe $\mathrm{cof}(\beta) = \omega$ and $j\,``\,\beta = \beta$, and hence $j(\beta) = \beta$. Also, $\beta \in S_\alpha$ some α, so $\beta = j(\beta) \in j(S_\alpha)$ and in S^*, contradiction. $\square$

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An equivalent formulation of the theorem is:

Corollary

If $j : V \to M$ is a nontrivial elementary embedding in V, then V is not a set-forcing extension of M.

In other words, if $j : V \to M$, then M is not a ground of V.

Converse embeddings $j : V \to V[G]$

Theorem (Woodin)

In any set-forcing extension $V[G]$, there is no nontrivial elementary embedding $j : V \to V[G]$.

Proof.

Suppose $j : V \to V[G]$. Let $\kappa = \mathrm{cp}(j)$. We may find $\lambda \gg \kappa, |\mathbb{P}|$ with $j(\lambda) = \lambda$ (but more subtle than before), and hence $j(\lambda^+) = \lambda^+$ and $j(\lambda^{++}) = \lambda^{++}$. Note that Cof_{λ^+} is absolute between V and $V[G]$. Partition $\mathrm{Cof}_{\lambda^+} \lambda^{++} = \bigsqcup_{\alpha<\kappa} S_\alpha$ into stationary sets in V. Let $S^* = j(\vec{S})(\kappa)$, stationary subset of $\mathrm{Cof}_{\lambda^+} \lambda^{++}$ in $V[G]$, disjoint from every $j(S_\alpha)$. Let $C = \{\, \beta < \lambda^{++} \mid j\,"\,\beta \subseteq \beta \,\}$, club subset of λ^{++} in $V[G]$. So $\exists \beta \in S^* \cap C$. Since $\mathrm{cof}(\beta) = \lambda^+$, it follows $j(\beta) = \beta$. So $\beta \in S_\alpha$ some α and so $\beta = j(\beta) \in j(S_\alpha)$ and $\beta \in S^*$, a contradiction. $\square$

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No j between grounds

Both theorems are instances of the following:

Theorem

If M and N are set-forcing ground models of V, then there is no nontrivial elementary embedding $j : M \to N$.

In other words, if M and N have a common set-forcing extension $M[G] = N[H] = V$, then there is no $j : M \to N$ there.

Corollary (Suzuki 1998)

In no set-forcing extension $V[G]$ is there a nontrivial elementary embedding $j : V \to V$.

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In no set-forcing extension $V[G]$ is there a nontrivial elementary embedding $j : V \to V$.

Generalizing to stationary-correct

A class model $M \subseteq V$ is *stationary correct* to V at δ if every stationary subset of δ in M remains stationary in V.

Theorem

If $M, N \models \mathrm{ZFC}$ are eventually stationary-correct to V, then there is no nontrivial elementary embedding $j : M \to N$.

The proof is to push harder on the previous arguments, and pay attention to some delicate details, but it works out. Essentially: suppose $j : M \to N$; find very large λ with $j(\lambda) = \lambda$. Partition $\mathrm{Cof}_{\lambda^+} \lambda^{++}$ into stationary sets in M. Let $S^* = j(\vec{S})(\kappa)$ stationary in N. Find $\beta = j(\beta)$ in both $j(S_\alpha)$ and in S^*.

There is no nontrivial elementary $j : V \to \mathrm{HOD}$

Theorem (Woodin)

There is no nontrivial elementary embedding $j : V \to \mathrm{HOD}$.

Proof.

Suppose $j : V \to \mathrm{HOD}$. Find $\lambda > \kappa = \mathrm{cp}(j)$ with $j(\lambda) = \lambda$, $j(\lambda^+) = \lambda^+$. Partition $\mathrm{Cof}_\omega \lambda^+ = \bigsqcup_{\alpha<\lambda^+} S_\alpha$ into stationary sets. Let $\vec{T} = j(\vec{S})$.

Claim: $\xi \in \mathrm{ran}(j)$ iff T_ξ is stationary in V. If $\xi = j(\alpha)$, then S_α and $j(S_\alpha) = T_{j(\alpha)}$ agree on club $C = \{\, \beta < \lambda^+ \mid j\,``\,\beta \subseteq \beta \,\}$. Conversely, if T_ξ stationary, then $\exists \beta \in C \cap T_\xi$, and so $\beta \in S_\alpha$ some α and $\beta = j(\beta) \in j(S_\alpha) = T_{j(\alpha)}$. So $\xi = j(\alpha)$.

Thus, $j\,``\,\lambda^+ \in \mathrm{HOD}$. Consequently, $C \in \mathrm{HOD}$. Complete argument as before: $\exists \beta \in C \cap T_\kappa$, but $\beta \in S_\alpha$ some α and $\beta = j(\beta) \in j(S_\alpha)$ and in T_κ, a contradiction. $\square$

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Claim: $\xi \in \mathrm{ran}(j)$ iff T_ξ is stationary in V. If $\xi = j(\alpha)$, then S_α and $j(S_\alpha) = T_{j(\alpha)}$ agree on club $C = \{\, \beta < \lambda^+ \mid j \,"\, \beta \subseteq \beta \,\}$. Conversely, if T_ξ stationary, then $\exists \beta \in C \cap T_\xi$, and so $\beta \in S_\alpha$ some α and $\beta = j(\beta) \in j(S_\alpha) = T_{j(\alpha)}$. So $\xi = j(\alpha)$.

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Theorem

If $j : V \to M$, then $V = \mathrm{HOD}(M)$.

Proof.

Suppose $j : V \to M$. Find large λ with $j(\lambda) = \lambda$, $j(\lambda^+) = \lambda^+$. Partition $\mathrm{Cof}_\omega \lambda^+ = \bigsqcup_{\alpha<\lambda^+} S_\alpha$ into stationary sets. Observe again $\xi \in \mathrm{ran}(j)$ if and only if T_ξ is stationary in V. Thus, $j \restriction \lambda^+$ is definable in V from $\vec{T}$, which is an element of M. Since any $A \subseteq \gamma$ is definable from $j(A)$ and $j\,"\,\gamma$, it follows that $A \in \mathrm{HOD}(M)$, and hence $V = \mathrm{HOD}(M)$. □

The theorem shows for any $j : V \to M$ that $j \restriction A$ is definable in V using parameters from M.

Corollary

If $j : V \to M$ and $M \subseteq \mathrm{HOD}$, then $V = \mathrm{HOD}$.

An improved version

The methods can be pushed to attain the following:

Theorem

If $j : M \to N$ and M is eventually stationary correct to V, then $M \subseteq \mathrm{HOD}(N)$ and $j \restriction A \in \mathrm{HOD}(N)$ for any $A \in M$.

Corollary

There is no generic embedding $j : V \to \mathrm{HOD}$. That is, in no set-forcing extension $V[G]$ is there a nontrivial elementary embedding $j : V \to \mathrm{HOD}^V$.

An improved version

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Consider the iterated HODs

One may attempt naively to iterate the HOD construction:

- $\mathrm{HOD}^0 = V$
- $\mathrm{HOD}^{n+1} = \mathrm{HOD}^{\mathrm{HOD}^n}$
- $\mathrm{HOD}^\omega = \bigcap_{n<\omega} \mathrm{HOD}^n$.

But there are meta-mathematical complications. In fact, we have no *uniform* definition of the HOD^n.

Harrington (1974), with related work of McAloon, shows consistent that every HOD^n can exist, but HOD^ω is not a class.

But some models have structure allowing a uniform account. Define that "HOD^η exists" to mean that we have a class H for which $H^0 = V$, $H^{\alpha+1} = \mathrm{HOD}^{H^\alpha}$ and $H^\gamma = \bigcap_{\alpha<\gamma} H^\alpha$ up to η.

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There is little reason to expect HOD^η definable, even when η is.

There is no $j : V \to \mathrm{HOD}^\eta$

Corollary

Assume HOD^η exists.

1. *There is no $j : V \to \mathrm{HOD}^\eta$.*
2. *If $M \subseteq \mathrm{HOD}^\eta$ is eventually stationary correct to HOD^η, then there is no $j : V \to M$.*
3. *Indeed, no such j exists in any $V[G]$.*

Proof.

For 1, we've already done the work: if $j : V \to \mathrm{HOD}^\eta$, then by the previous $V = \mathrm{HOD}(\mathrm{HOD}^\eta)$ and so $V = \mathrm{HOD}$ and so $\mathrm{HOD}^\eta = V$. So we reduce to $j : V \to V$, a contradiction. □

Consider the generic HOD

The *generic* HOD, introduced by Fuchs, is the intersection of the HODs of all forcing extensions.

$$\mathrm{gHOD} = \bigcap \mathrm{HOD}^{V[G]}$$

Collapse forcing suffices. $\mathrm{gHOD} \models \mathrm{ZFC}$, and is invariant by set forcing. The gHOD can be far smaller than HOD and also than the *mantle*, the intersection of all grounds.

Theorem

If $j : V \to N$, then $V = \mathrm{gHOD}(N)$. If $j : M \to N$ and M is eventually stationary correct to V, then $M \subseteq \mathrm{gHOD}(N)$ and $j \restriction A \in \mathrm{gHOD}(N)$ every $A \in M$.

Corollary

If $j : V \to M$ and $M \subseteq \mathrm{gHOD}$, then $V = \mathrm{gHOD}$.

There is no $j : V \to \mathrm{gHOD}$

Corollary

1. *There is no nontrivial elementary $j : V \to \mathrm{gHOD}$.*
2. *If gHOD^η exists, then there is no $j : V \to \mathrm{gHOD}^\eta$.*
3. *If $M \subseteq \mathrm{gHOD}^\eta$ is eventually stationary correct to gHOD^η, then there is no $j : V \to M$.*

Corollary

For any class A, if $\mathrm{HOD}[A]^\eta$ exists, then there is no $j : V \to \mathrm{HOD}[A]^\eta$ and no $j : V \to M$ for any $M \subseteq \mathrm{HOD}[A]^\eta$ eventually stationary correct to $\mathrm{HOD}[A]^\eta$. Similarly for $\mathrm{gHOD}[A]^\eta$. And no such j exists in any $V[G]$.

There is no $j : V \to \mathrm{gHOD}$

Corollary

1. *There is no nontrivial elementary* $j : V \to \mathrm{gHOD}$.
2. *If* gHOD^η *exists, then there is no* $j : V \to \mathrm{gHOD}^\eta$.
3. *If* $M \subseteq \mathrm{gHOD}^\eta$ *is eventually stationary correct to* gHOD^η*, then there is no* $j : V \to M$.

Corollary

For any class A*, if* $\mathrm{HOD}[A]^\eta$ *exists, then there is no* $j : V \to \mathrm{HOD}[A]^\eta$ *and no* $j : V \to M$ *for any* $M \subseteq \mathrm{HOD}[A]^\eta$ *eventually stationary correct to* $\mathrm{HOD}[A]^\eta$*. Similarly for* $\mathrm{gHOD}[A]^\eta$*. And no such* j *exists in any* $V[G]$.

Embeddings $j : \mathrm{HOD} \to V$

Let me turn now to the case of embeddings $j : \mathrm{HOD} \to V$ and other definable classes.

The arguments will have a very different character, and will not rely on any result in infinite combinatorics, such as Solovay's stationary partition result.

Instead, we shall extend the embedding $\mathrm{HOD} \to V$ into an infinite inverse system of embeddings

$$\cdots \longrightarrow \mathrm{HOD}^n \longrightarrow \cdots \longrightarrow \mathrm{HOD}^2 \longrightarrow \mathrm{HOD} \longrightarrow V,$$

and then analyze the nature of the inverse limit. The overall argument is soft, but details run into subtle meta-mathematical issues, which we resolve.

The Kunen inconsistency under $V = \mathrm{HOD}$

There is a very easy proof when $V = \mathrm{HOD}$:

Theorem

If $V = \mathrm{HOD}$, then there is no nontrivial $j : V \to V$.

Proof.

Let $\lambda = \sup\langle \kappa_n \mid n < \omega\rangle$. It follows that $j(\lambda) = \lambda$. Let $s = \langle \alpha_n \rangle_n$ be HOD-least ω-sequence with $\lambda = \sup(s)$. Since s is definable from λ, it follows that $j(s) = s$ and hence also $j(\alpha_n) = \alpha_n$. But j has no fixed points between κ and λ, a contradiction. □

The argument needs only a definable well-ordering of $[\lambda]^\omega$.

There is no $j : \mathrm{HOD} \to V$

Theorem (Hamkins,Kirmayer,Perlmutter)

There is no nontrivial elementary embedding $j : HOD \to V$.

Proof

Suppose $j : \mathrm{HOD} \to V$. Extend to an inverse system

$$\cdots \longrightarrow \mathrm{HOD}^n \longrightarrow \cdots \longrightarrow \mathrm{HOD}^2 \longrightarrow \mathrm{HOD} \longrightarrow V$$

The subtle issue about uniform presentation of HOD^n is resolved by proving $\mathrm{HOD}^n = \mathrm{dom}(j^n)$. Let $<^{n+1}$ be the canonical well-ordering of HOD^{n+1} definable in HOD^n. Let $<^0 = j(<^1)$. Thus, $j(<^{n+1}) = <^n$. Define $\vec{x} = \langle x_n \mid n < \omega \rangle$ is *j-coherent*, if $j(x_{n+1}) = x_n$ for all $n < \omega$. ...

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$$\cdots \longrightarrow \mathrm{HOD}^n \longrightarrow \cdots \longrightarrow \mathrm{HOD}^2 \longrightarrow \mathrm{HOD} \longrightarrow V$$

Claim. Every j-coherent sequence is constant.

Proof: Suppose $\vec{x} = \langle x_n \mid n < \omega\rangle$ is nonconstant j-coherent, with $\in$-minimal x_0. It follows all x_n same rank. Also, $x_{n+1} \neq x_n$. Let a_n be $<^n$-least in $x_{n+1} \bigtriangleup x_n$. It follows $\langle a_n \mid n < \omega\rangle$ is j-coherent, nonconstant, lower rank, a contradiction.

Claim. There is a non-constant j-coherent sequence.

Proof: Let y_n be the $<^n$-least element of $\mathrm{HOD}^n \setminus \mathrm{HOD}^{n+1}$. It follows by the j-coherence of the relations $<^n$ that $j(y_{n+1}) = y_n$, and so this sequence is j-coherent. Since $y_0 \in V \setminus \mathrm{HOD}$ and $y_1 \in \mathrm{HOD} \setminus \mathrm{HOD}^2$, it follows that $y_0 \neq y_1$, and so this sequence is not constant.

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A generalization

Theorem

Suppose $j : M \to N$, where $M \subseteq N \models \mathrm{ZF}$ and M is b-definable in N with $j(b) = b$, and A is b-definable class in N with tcl(A) has b-definable well-ordering in N. Then $A^M = A^N$.

The proof uses a similar idea, expanding to inverse system

$$\cdots \longrightarrow M^n \longrightarrow \cdots \longrightarrow M^2 \longrightarrow M^1 \longrightarrow M^0,$$

and considers j-coherent sequences, establishing first that they are all constant, and second, under the assumption that $A^M \neq A^N$, that there is a nonconstant j-coherent sequence.

A surprising level of agreement

Corollary

If $j : M \to N$ is elementary for $M \subseteq N \models \mathrm{ZF}$ and M is b-definable in N for parameter $b = j(b)$, then M and N have

1. *the same cardinals and the same cofinality function,*
2. *the same continuum function,*
3. *the same $\diamondsuit_\kappa^*$ pattern and*
4. *and the same large cardinals of any particular kind.*
5. *Furthermore, $\mathrm{HOD}^M = \mathrm{HOD}^N$ and $\mathrm{gHOD}^M = \mathrm{gHOD}^N$ and more.*

There is no $j : \mathrm{HOD}^2 \to \mathrm{HOD}$

Corollary

If $M \subsetneq N \models \mathrm{ZF}$, with M definable in N and $M \subseteq \mathrm{HOD}^N$, then there is no $j : M \to N$.

Proof.

If such $j : M \to N$, then $\mathrm{HOD}^M = \mathrm{HOD}^N$. Thus, $M \subseteq \mathrm{HOD}^N = \mathrm{HOD}^M \subseteq M$ and so $M = \mathrm{HOD}^M$, and consequently $N = \mathrm{HOD}^N$ and so $M = N$, contradiction. □

Corollary

There is no $j : \mathrm{HOD}^2 \to \mathrm{HOD}$, if different, and no $j : \mathrm{HOD}^n \to \mathrm{HOD}^m$ for $m < n$, if different. Similarly no $j : \mathrm{gHOD}^2 \to \mathrm{gHOD}$ etc.

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A sweeping general result

The proof method leads to a sweeping result:

Theorem

If M is a definable transitive class, then there is no nontrivial elementary embedding $j : M \to V$.

This is a GBC scheme. The nonexistence of $j : \mathrm{HOD} \to V$ is a special case, generalizing to no $j : \mathrm{HOD}^n \to V$.

The theorem is a consequence of the following general formulation.

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The theorem is a consequence of the following general formulation.

Definable domain

Theorem

If $j : M \to N$ and $M \subseteq N$ and N is eventually stationary correct to V, then M is not definable in N from parameters fixed by j.

The proof uses the stationary partition methods we saw earlier, but making critical use of the fact that if $j : M \to N$ and M is definable in N via parameters fixed by j, then $\mathrm{Cof}^M_\omega = \mathrm{Cof}^N_\omega$.

Corollary

If M is a definable class in V, then in no set-forcing extension $V[G]$ is there a nontrivial elementary $j : M \to V$.

For example, there is no generic $j : \mathrm{HOD} \to V$ or $j : \mathrm{gHOD} \to V$.

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Nontrivial $j : \mathrm{HOD} \to \mathrm{HOD}$?

It is open whether there can be $j : \mathrm{HOD} \to \mathrm{HOD}$.

The following corollary may be a way to produce such j.

Corollary

Do not assume AC. If $j : M \to V$ is a nontrivial elementary embedding from a transitive proper class M that is definable in V from parameters fixed by j, then there is a nontrivial elementary embedding from HOD to HOD.

Proof.

By earlier theorem, $\mathrm{HOD}^M = \mathrm{HOD}^V$, and so $j \upharpoonright \mathrm{HOD} : \mathrm{HOD} \to \mathrm{HOD}$ is the desired embedding. □

Definable embeddings

Let's turn now to the case where the embedding j is not merely a GBC class, but a first-order definable class (with parameters).

In this case, many of the arguments admit of soft proofs, requiring neither any results from infinite combinatorics nor the axiom of choice.

Definable embeddings

An embedding $j : M \to N$ is definable in V using parameter p, when there has been provided a first-order formula $\varphi(x, y, z)$, such that $j(x) = y$ if and only if $\varphi(x, y, p)$ holds in V.

For a given formula φ, the question whether a given parameter p succeeds in $\varphi(\cdot, \cdot, p)$ defining a nontrivial elementary embedding $j : V \to V$ is a first-order expressible property of p.

Similarly, for a given formula φ, the collection of ordinals κ which arise as the critical point of such an embedding is definable.

The Kunen inconsistency for definable j

The case of definable embeddings is easy to refute:

Theorem (Folklore, Suzuki)

Assume only ZF*. There is no nontrivial elementary embedding $j : V \to V$ that is definable from parameters.*

Proof.

Suppose $j(x) = y \iff \varphi(x, y, p)$. Choose p so that the resulting critical point κ is as small as possible, using this φ. So $j(\kappa)$ is also like that, a contradiction. $\square$

This is essentially the same as the classical observation that Reinhardt cardinals are not first-order.

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This is essentially the same as the classical observation that Reinhardt cardinals are not first-order.

Generic definable embeddings

Theorem

Do not assume AC. There is no $j : M \to V$ definable in any set-forcing extension $V[G]$, allowing $M \subseteq V[G]$.

Proof.

Suppose $j : M \to V$ is defined in $V[G]$ by $\varphi(\cdot, \cdot, b)$. So $\exists q \in G$ forcing $\varphi(\cdot, \cdot, \dot{b})$ defines an embedding. Assume without loss that κ is smallest possible critical point arising this way using φ, any $\mathbb{Q}$. So κ is definable in V without parameters. But $j : M \to V$ is elementary and $\kappa \notin \mathrm{ran}(j)$, a contradiction. $\square$

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Definable generic embeddings

We immediately deduce the following as special cases:

Corollary

Do not assume AC.

1. *There is no $j : M \to V$ definable with parameters in V.*
2. *There is no $j : V \to V$ definable with parameters in $V[G]$.*
3. *There is no $j : V[G] \to V$ definable with parameters in $V[G]$.*
4. *There is no $j : V \to V[G]$ definable with parameters in $V[G]$.*

There is no definable $j : \mathrm{HOD} \to \mathrm{HOD}$

Theorem

Do not assume AC. *There is no nontrivial $j : \mathrm{HOD} \to \mathrm{HOD}$ definable in V from parameters.*

Proof.

Formally a ZF scheme. Suppose $j : \mathrm{HOD} \to \mathrm{HOD}$ defined by $j(x) = y \iff V \models \varphi(x, y, b)$. (Perhaps $b \notin \mathrm{HOD}$.) Let $\kappa = \mathrm{cp}(j)$. By reflection, there is a definable class club C of γ with φ and $\exists y \varphi(x, y, z)$ absolute V_γ to V. So $\gamma \in C \implies j\,"\,\gamma \subseteq \gamma$. Let $\delta = \omega^{\text{th}}$ in C above κ and $\rho(b)$. In particular, $j\,"\,\delta \subseteq \delta$ and $\mathrm{HOD} \models \mathrm{cof}(\delta) = \omega$, and so $j(\delta) = \delta$ and hence $j((\delta^+)^{\mathrm{HOD}}) = (\delta^+)^{\mathrm{HOD}}$. Let $\gamma = (\delta^+)^{\mathrm{HOD}}$-th element of C. So $j\,"\,\gamma \subseteq \gamma$ and $\mathrm{HOD} \models \mathrm{cof}(\gamma) = (\delta^+)^{\mathrm{HOD}}$, and so $j(\gamma) = \gamma$.

This is now enough to run the stationary-partition argument using $(\mathrm{Cof}_\omega\, \gamma)^{\mathrm{HOD}} = \bigsqcup_{\alpha<\kappa} S_\alpha$ etc. □

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No definable $j : \mathrm{HOD} \to \mathrm{HOD}$

Note that the proof that there is no definable $j : \mathrm{HOD} \to \mathrm{HOD}$ is much simpler in the case where j is definable without parameters or with ordinal parameters, for in this case one gets directly that $j \restriction \theta \in \mathrm{HOD}$ for every ordinal θ, and this is enough to complete the argument.

Indeed, when $j : \mathrm{HOD} \to \mathrm{HOD}$ is definable in V using no parameters or using ordinal parameters, then HOD satisfies $\mathrm{ZFC}(j)$ and so we have directly an instance of the Kunen inconsistency by restricting to $\langle \mathrm{HOD}, \in, j \rangle$.

But the full proof treats the case j is definable using parameters not necessarily in HOD.

Open questions

The main open question in this area is whether the Kunen inconsistency requires AC.

Question

Is it consistent with $\mathrm{ZF}(j)$ that $j : V \to V$ is a nontrivial elementary embedding of the universe to itself?

We are naturally also interested in the corresponding question for each of the generalizations of the Kunen inconsistency whose current proofs use AC.

For example, in the $\neg\mathrm{AC}$ context, can there be nontrivial elementary embeddings $j : V[G] \to V$ or $j : V \to V[G]$ for a set-forcing extension $V[G]$?

Can there be $j : \mathrm{HOD} \to \mathrm{HOD}$?

The second main question is:

Question

Is it consistent that there is a nontrivial elementary embedding $j : \mathrm{HOD} \to \mathrm{HOD}$?

We ask in the GBC context, but it is also sensible to drop AC.

There are numerous other questions. To what extent do the theorems we have mentioned about embeddings arising in set-forcing extensions also apply to class forcing? Or to certain kinds of class forcing? Or to other non-forcing extensions? To what extent do the theorems on HOD generalize to other natural definable classes?

Thank you.

Preprint available at http://arxiv.org/abs/1106.1951

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