Lecture 7. Iteration trees of transfinite length.

It is often important to continue iterating into the transfinite. The way we continue a tree $\mathcal{T}$ of limit length λ is: pick a branch b which has been visited cofinally often below λ, and such that $\mathcal{M}_b^{\mathcal{T}}$ is wellfounded. Set $\mathcal{M}_\lambda^{\mathcal{T}} = \mathcal{M}_b^{\mathcal{T}}$, and continue. To be more precise:

Definition 8. Let $\gamma \in \mathrm{ORD}$. We call T a tree order on γ iff

(1) T is a strict partial order of γ,
(2) $\forall \beta < \gamma$ (T wellorders $\{\alpha \mid \alpha T \beta\}$),
(3) $\forall \alpha, \beta$ ($\alpha T \beta \Rightarrow \alpha < \beta$),
(4) $\forall \alpha$ ($0 < \alpha \rightarrow 0 T \alpha$),
(5) $\forall \alpha$ (α is a successor ordinal iff α is a T-successor), and
(6) $\forall \lambda < \gamma$ (λ is a limit ordinal $\Rightarrow \{\alpha \mid \alpha T \lambda\}$ is $\in$-cofinal in λ).

<u>Definition 2</u> Let $\gamma \in ORD$. A <u>nice iteration tree of length γ on M</u> is a system $\mathcal{T} = \langle T, \langle (M_\alpha, E_\alpha) \mid \alpha < \gamma \rangle, \langle i_{\alpha\beta} \mid \alpha T \beta \rangle \rangle$ such that

(1) T is a tree order on γ,

(2) $M_0 = M$,

(3) $M_\alpha \models E_\alpha$ is a nice extender,

(4) $\alpha < \beta \Rightarrow lh(E_\alpha) < lh(E_\beta)$

(5) if $\alpha + 1 < \gamma$, then

$M_{\alpha+1} = Ult(M_\xi, E_\alpha)$, where ξ is least s.t. $M_\xi \frown_{crit(E_\alpha)+1} M_\alpha$

and

$i_{\xi,\alpha+1}$ = canonical embedding, and

$i_{\nu,\alpha+1} = i_{\xi,\alpha+1} \circ i_{\nu,\xi}$ for $\nu T \xi$,

(6) if $\lambda < \gamma$ is a limit ordinal

$M_\lambda = \text{dir lim}_{\alpha T \lambda} M_\alpha$

$i_{\alpha\lambda}$ = canonical embedding, for $\alpha T \lambda$.

Is it always possible to continue a nice iteration tree on V? At successor steps, yes.

__Theorem 3__ Let $M \models ZFC$ be transitive, and closed under ω-sequences. Let $\mathcal{T}$ be a nice iteration tree on M of length $\alpha+1$, and let $\xi \leq \alpha$ be such that $\mathcal{M}^{\mathcal{T}}_{\xi} \underset{\mathrm{crit}(E^{\mathcal{T}}_{\alpha})+1}{\frown} \mathcal{M}^{\mathcal{T}}_{\alpha}$. Then $\mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\xi}, E^{\mathcal{T}}_{\alpha})$ is wellfounded.

__Proof__. We ~~also~~ need the following exercise:

__Exercise 33__. Let $\mathcal{T}$ be a nice iteration tree, and $\alpha < \mathrm{lh}(\mathcal{T})$. Show

$$\mathcal{M}^{\mathcal{T}}_{\alpha} = \{ i^{\mathcal{T}}_{0\alpha}(f)(a) \mid f \in \mathcal{M}^{\mathcal{T}}_{0} \wedge a \in [\nu]^{<\omega} \},$$

where

$$\nu = \sup\{ \mathrm{lh}(E^{\mathcal{T}}_{\xi}) \mid (\xi+1) \, T \, \alpha \text{ or } \xi+1 = \alpha \}$$

The exercise generalizes the fact that

$$Ult(M,E) = \{ i^M_E(f)(a) \mid a\, f \in M \wedge a \in [lh E]^{<\omega} \}.$$

It says that $\mathcal{M}^{\mathcal{T}}_\alpha$ is Skolem-generated by $\operatorname{ran}(i^{\mathcal{T}}_{0\alpha})$ together with ordinals below the sup of the lengths of extenders used on the branch 0-to-α.

Now let ξ, α be as in the ~~th~~ theorem, and set

$$N = Ult(M^{\mathcal{T}}_\xi, E^{\mathcal{T}}_\alpha).$$

Let

$$i : M \longrightarrow N$$

be the canonical emb. ($i = \pi \circ i^{\mathcal{T}}_{0\xi}$, where $\pi : M^{\mathcal{T}}_\xi \to N$.) Let $\lambda = lh(E^{\mathcal{T}}_\alpha)$. The proof of exercise 33 easily yields

$$N = \{ i(f)(a) \mid a \in [\lambda]^{<\omega} \wedge f \in M \}.$$

Note here that although N may be illfounded,

$$i(\kappa) \in \mathrm{wfp}(N),$$

where $\kappa = \mathrm{crit}(E_\alpha^{\bar{\mathcal{T}}})$. This is because

$$V_{i(\kappa)+1}^{N} = V_{i(\kappa)+1}^{\mathrm{Ult}(M_\alpha^{\mathcal{T}}, E_\alpha^{\mathcal{T}})},$$ and ~~the~~

$V_{i(\kappa)+1}^{\mathrm{Ult}(M_\alpha^{\mathcal{T}}, E_\alpha^{\mathcal{T}})}$ can be computed in $M_\alpha^{\bar{\mathcal{T}}}$, which thinks $E_\alpha^{\bar{\mathcal{T}}}$ is an extender.

Now pick $\langle f_k \mid k \in \omega \rangle$ such that there are $a_k \in [\lambda]^{<\omega}$ with

$$i(f_{k+1})(a_{k+1}) \in^N i(f_k)(a_k)$$

for all k. Note $\langle f_k \mid k \in \omega \rangle \in M$!

Thus

$$i(\langle f_k \mid k \in \omega \rangle) = \langle i(f_k) \mid k \in \omega \rangle \in N.$$

Now pick γ s.t.

$$N \models \gamma \in \mathrm{ORD} \wedge \lambda < \gamma \wedge \langle i(f_k) \mid k \in \omega \rangle \in V_\gamma.$$

Working in N, we have H, π such that

$$N \models H \text{ is transitive}, \ |H| = \lambda,$$
$$\text{and } \pi : H \to V_\gamma \text{ with}$$
$$\pi \restriction \lambda = \mathrm{id} \text{ and } \langle i(f_k) \mid k \in \omega \rangle \in \mathrm{ran}(\pi).$$

Now $(\mathrm{ran}(\pi), \in^N)$ is illfounded in V, hence $(H, \in^N)$ is illfounded in V. But $H \in V^N_{i(\kappa)} \subseteq \mathrm{wfp}(N)$, a contradiction.

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Insofar as continuing nice trees on V at limit steps goes, the main result is the following.

<u>Theorem 4</u>. Let $\mathcal{T}$ be a nice iteration tree on V of countable limit length λ. Suppose that for all limit $\eta < \lambda$, $\{\alpha \mid \alpha T \eta\}$ is the <u>unique</u> cofinal wellfounded branch of $\mathcal{T} \restriction \eta$. Then $\mathcal{T}$ has a cofinal, wellfounded branch.

<u>Remark</u> In other words, if $\mathcal{T}$ has made the only choice it could make at limit $\eta < \lambda$, then there is a choice for it to make at λ.

We shall sketch the proof of Theorem 4 in an appendix to this lecture. It is much like the proof of Theorem 6.9.

This leads us to one of the biggest open problems in the subject.

<u>Definition 5</u> <u>nice-UBH</u> is the statement: Every nice iteration tree on V of limit length has at most one cofinal, wellfounded branch.

Definition 6 generic-nice-UBH is the statement: $V[G] \models$ nice-UBH, whenever G is set-generic over V.

Whether nice-UBH, or better generic-nice-UBH, are true are very important questions. Of course, the more useful answer would be "yes". The reason is that we would then get, via theorem 4, an iteration strategy for V. We now explain that concept more precisely.

Let $M \models ZFC$ be transitive, and $\theta \in ORD$. The (nice) iteration game of length θ on M is played as follows: there are two players, I and II. They cooperate to produce an iteration tree $\mathcal{T}$ on M. At successor rounds

$\alpha+1$, player I extends $\mathcal{T}$ by picking a nice $E_\alpha^{\mathcal{T}}$ from $\mathcal{M}_\alpha^{\mathcal{T}}$, and setting $\mathcal{M}_{\alpha+1}^{\mathcal{T}} = \mathrm{Ult}(M_\xi^{\mathcal{T}}, E_\alpha^{\mathcal{T}})$ for ξ least such that $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \mathrm{lh}(E_\xi^{\mathcal{T}})$. (If the ultrapower is illfounded, the game ends, and I has won.) At limit rounds $\lambda < \theta$, II extends $\mathcal{T}$ by picking b cofinal in λ such that $\mathcal{M}_b^{\mathcal{T}}$ is wellfounded. If ~~he~~ II fails to do this, I wins.

If after θ rounds, I has not yet won, then II wins.

We call this game $G_{\text{nice}}(M, \theta)$. A winning strategy for II in $G_{\text{nice}}(M, \theta)$ is called a <u>θ-iteration strategy</u> for M. We say <u>M is θ-iterable</u> (for nice trees) iff there is ~~such~~ a θ-iteration strategy for M.

We have

<u>Theorem 7</u>. If nice-UBH holds, then V is ω_1-iterable for nice trees.

<u>Proof</u>. Player II's strategy in $G_{nice}(V, \omega_1)$ is: at round λ, pick the unique cofinal wellfounded branch of $\mathcal{T} \upharpoonright \lambda$.

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ω_1+1 iterability is much more useful than ω_1 iterability. We have

<u>Theorem 8</u> If generic-nice-UBH holds, then V is κ-iterable for nice trees, where κ is the least measurable cardinal.

<u>Exercise 34</u> Prove theorem 8. [You need to know something about preservation of large cardinals under small forcing to do

this one, so it's really only for the more advanced students.]

The main result known in the direction of proving UBH is the following. For $\mathcal{T}$ a nice tree, set

$$\delta(\mathcal{T}) = \sup\{lh(E_\alpha^{\mathcal{T}}) \mid \alpha < lh(\mathcal{T})\},$$

$$\mathcal{M}(\mathcal{T}) = \bigcup_{\alpha < lh(\mathcal{T})} V^{M_\alpha^{\mathcal{T}}}_{lh E_\alpha^{\mathcal{T}}}.$$

So $ORD \cap \mathcal{M}(\mathcal{T}) = \delta(\mathcal{T})$, and

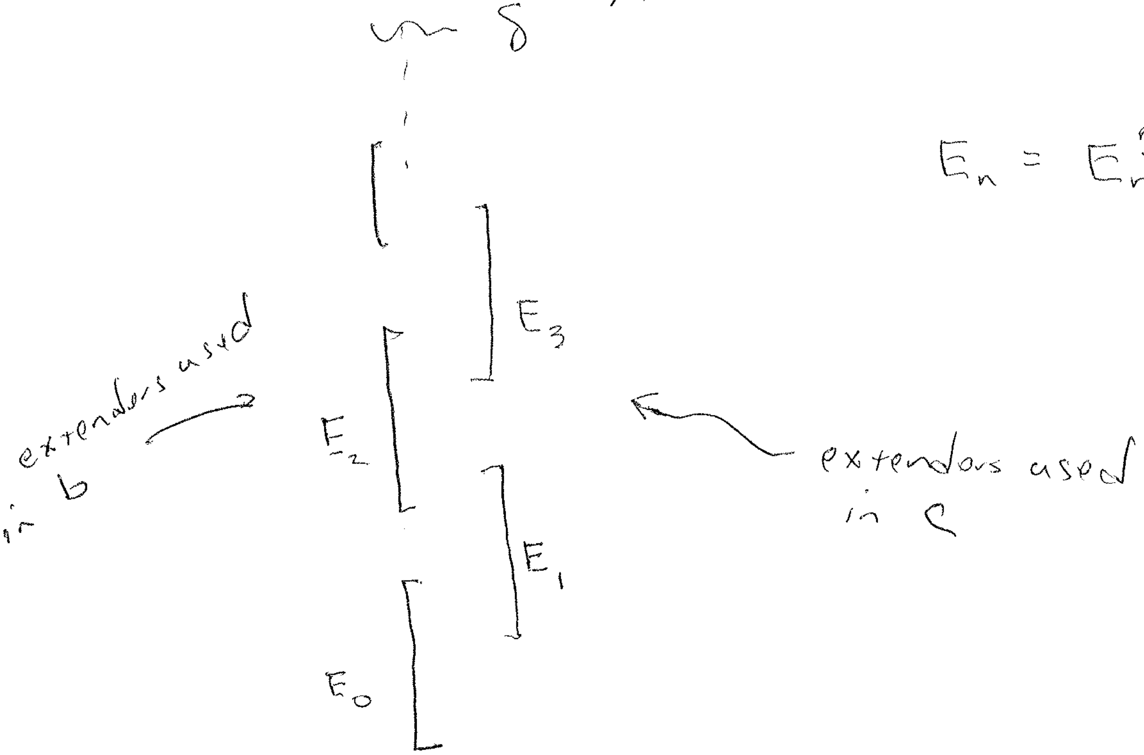
$\mathcal{M}(\mathcal{T}) = V^{M_b^{\mathcal{T}}}_{\delta(\mathcal{T})}$ for any cofinal branch b of $\mathcal{T}$ such that $\delta(\mathcal{T}) \in M_b^{\mathcal{T}}$.

Theorem 9 Let $\mathcal{T}$ be a nice iteration tree of limit length, $\delta = \delta(\mathcal{T})$, and suppose b and c are distinct cofinal branches of $\mathcal{T}$ such that $\delta \in M_b^{\mathcal{T}} \cap M_c^{\mathcal{T}}$. Let $A \subseteq \delta$, and $A \in M_b^{\mathcal{T}} \cap M_c^{\mathcal{T}}$. Then

$$(M(\mathcal{T}), \in, A) \models \text{" } \exists \kappa \, (\kappa \text{ is } A\text{-reflecting in ORD}) \text{"}.$$

Remark Another way this is often stated is: $\delta(\mathcal{T})$ is Woodin with respect to all $A \in M_b^{\mathcal{T}} \cap M_c^{\mathcal{T}}$, with respect to extenders in $M(\mathcal{T})$.

Proof of theorem 9. (Sketch.) Let us consider the special case $\mathcal{T}$ is an alternating chain, and b is its even branch, and c is its odd branch. Note that the extenders of $\mathcal{T}$ overlap in the following "zipper" pattern



$E_n = E_n^{\mathcal{T}}$

That is, letting $\kappa_n = \mathrm{crit}(E_n)$ and $\lambda_n = \mathrm{lh}(E_n)$:

$$\kappa_n < \kappa_{n+1} < \lambda_n$$

for all n. Now let $A \subseteq \delta$ and $A \in M_b \cap M_c$. Pick m large enough that

$$A \in \mathrm{ran}(i_{m,b}) \cap \mathrm{ran}(i_{m+1,c}).$$

Claim For any $n \geq m$,

$$i_{E_n}(A \cap \kappa_n) \cap \lambda_n = A \cap \lambda_n.$$

(That is, i_{E_n} shifts A to itself below the next critical point. It doesn't matter whether we write $i^{m(\bar{x})}_{E_n}$ or $i^{M^a_{n-1}}_{E_n}$ here, since the ultrapowers are the same below the image of κ_n.)

Proof Suppose e.g. $n+1 \in b$. Let $A = i_{m,b}(\bar{A})$. Then $i_{m,n-1}(\bar{A}) \cap \kappa_n = A \cap \kappa_n$, because $\operatorname{crit}(i_{n+1,b}) = \kappa_n$.

So $i_{n-1,b}(A \cap \kappa_n)$ agrees with A below $i_{n-1,b}(\kappa_n)$. But $i_{n-1,b}(A \cap \kappa_n)$ agrees with $i_{E_n}(A \cap \kappa_n)$ below λ_n, because $\operatorname{crit} i_{n+1,b} \geq \lambda_n$.

Consider the ~~extender representing~~ embedding

$$i_{E_n \restriction \kappa_{n+1}} \circ \cdots \circ i_{E_{m+1} \restriction \kappa_{m+2}} \circ i_{E_m \restriction \kappa_{m+1}} = j$$

Note $E_i \restriction \kappa_{i+1} \in M(\mathcal{J})$, because $\kappa_{i+1} < lh(E_i)$! It is routine to show that j witnesses κ_m is A-reflecting to β in $M(\mathcal{J})$.

<u>Exercise 35</u> (a) Complete the proof of this.

(b) Complete the proof of Theorem 9 as follows: let b and c be distinct cofinal branches of $\mathcal{J}$. Find $\langle \alpha_n \mid n \in \omega \rangle$ cofinal in λ such that

$\alpha_{2n} \in b$,

$\alpha_{2n+1} \in c$,

$$\alpha_{2n}+1 \in b,$$

$$\alpha_{2n+1}+1 \in C,$$

and

$$\mathrm{crit}(E_{\alpha_k}) < \mathrm{crit}(E_{\alpha_{k+1}}) < \mathrm{lh}(E_{\alpha_k})$$

for all k. I.e. we have the zipper pattern embedded in the two branches. Now argue as above.

Remark For more detail, see "Iteration trees", Martin, Steel, JAMS.

Remark So if $\mathcal{T}$ has distinct cofinal branches, then $\mathrm{lh}(\mathcal{T})$ has cofinality ω. This is also easy to see from the fact that every branch of an iteration tree is closed below its sup (as a set of ordinals).

<u>Corollary 10</u> Suppose nice-UBH fails. Then there is a proper class model with a Woodin cardinal.

<u>Proof</u> Let $\mathcal{T}$ be nice on V, and have distinct cofinal wellfounded branches b and c. Then

$$L(M(\mathcal{T})) \subseteq M_b^{\mathcal{T}} \cap M_c^{\mathcal{T}}.$$

So by theorem 9,

$$L(M(\mathcal{T})) \models \delta(\mathcal{T}) \text{ is Woodin.}$$

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