A conjecture of Alon and Tarsi and a lemma by David Glynn

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Even and odd Latin Squares

Consider a latin square on symbols $1, 2, \ldots, n$. Every row defines a permutation in S_n, every column as well, and every symbol defines a permutationmatrix, hence a permutation. The row (column, symbol) sign of the square is the product of the signs of the row (column, symbol) permutations.
Here is a first exercise: the product of the three signs is $(-1)^{\binom{n}{2}}$.

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It is customary to call a latin square *even* if the product of the row and the column signs is $+1$, and odd otherwise. In this talk we prefer to use the symbol sign to distinguish even and odd latin squares.

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The following is known about this problem: The (stronger) conjecture is true for $n = p - 1$ (Glynn, 2010), $n = p$ (Drisko, 1998), and $n = p + 1$ (Drisko, 1997).

A ridiculous plan

Consider an $n \times n$ matrix X, whose entries are n^2 variables x_{ij}. Now $\det(X)^n$ is a polynomial of degree n^2, and the coefficient of $\prod_{ij} x_{ij}$ is precisely the difference of the number of even and the number of odd latin squares. So it suffices to show that this coefficient is nonzero.

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This I tried five or so years ago in all possible ways, but I didn't succeed.

If only I had read the literature for once!

Glynn's Lemma, from 1998

Here is Glynn's identity:
Let p be a prime, and X an $m \times m$ matrix with entries from $GF(p)$. Then modulo p:

$$\det{}_p(X) = \det(X)^{p-1}.$$

Here $\det{}_p(X) := (-1)^m \sum \frac{\mathbf{x}^{\mathbf{e}}}{\mathbf{e}!}$, where $\mathbf{x}^{\mathbf{e}} = x_{11}^{e_{11}} x_{12}^{e_{12}} \cdots x_{mm}^{e_{mm}}$ and $\mathbf{e}! = e_{11}!e_{12}! \cdots e_{mm}!$, and the sum is over all matrices $\mathbf{e}$ with nonnegative integer entries and with row and column sums equal to $p-1$.

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The easy proof: Show that $\det(X)\det{}_p(X) = \det(X)^p$.

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For some reason it took David Glynn 12 years to realize that his lemma proves Alon-Tarsi for $n = p - 1$.
His lemma gives that the coefficient of $\prod_{ij} x_{ij}$ in $\det(X)^{p-1}$ is 1 modulo p, so nonzero.
Recall that this coefficient is the difference between the number of even and odd latin squares.

Next consequence: Gen. Alon-Tarsi for $n = p$

If n is odd, then we look at latin squares with constant diagonal. In this case we look at the coefficient of $\prod_{i \neq j} x_{ij}$ in $\det(X)^{n-1}$ for our $n \times n$ matrix $X = (x_{ij})$ (say with zero diagonal, so $x_{ii} = 0$).

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Glynn's lemma tells us again that this coefficient is 1 mod p.

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Now take again our $n \times n$ matrix X, with zero diagonal, and also $x_{12} = 0$ and look at the coefficient of a square free monomial $\prod x_{ij}$ in $\det(X)^{p-1}$.

$n = p + 1$, continued

Glynn's lemma gives that this is one, where the matrix $\mathbf{e}$ has row and column sums $p - 1$ and zero diagonal, and hence additional zeros on a derangement matrix for a derangement sending 1 tot 2. So for every derangement the contribution to the number of even minus the number of odd squares is simply the sign of the derangement.
Since the number of even derangements minus the number of odd derangements is $(-1)^p = -1$, the same is true for latin squares (mod p).

Thank you David Glynn!