Measurement Matrices for Compressive Sensing via Column Replacement

Charles J. Colbourn[1], Daniel Horsley[2], and Violet R. Syrotiuk[1]

[1]Arizona State University
[2]Monash University

Workshop on Combinatorial Designs, 2011

Outline

Motivation

- Probabilistic algorithms to construct compressive sensing matrices do so with very high probability
  - But, how to check that all the properties are satisfied?
  - The analysis of such algorithms make assumptions on the random mechanism that may be difficult to implement in practice
- Our interest: the *deterministic* construction of measurement matrices

Compressive sensing

- A signal $\mathbf{x}$, which is a vector in $\mathbb{R}^k$, having at most t nonzero coordinates.
- A *sample* is a vector of weights $\mathbf{w} \in \mathbb{R}^k$, for which the sample measurement is $\mathbf{wx}^T$.
- Goal: Construct a set of N samples so that the unknown signal $\mathbf{x}$ can be recovered from the sample measurements. The $N \times k$ matrix so formed is a *measurement matrix*.
- (Admittedly, this is an overly simplified version of compressive sensing!)

Compressive Sensing

Recoverability

- A measurement matrix A *has* (ℓ_0, t)*-recoverability* when it permits exact recovery of $\mathbf{x}$ using $A\mathbf{x} = \mathbf{b}$, given A and $\mathbf{b}$, and the fact that $\mathbf{x}$ is t-sparse.
- A measurement matrix A *has* (ℓ_1, t)*-recoverability* when, for each t-sparse signal $\mathbf{x}$, $\mathbf{x}$ is the unique solution to $\min\{||\mathbf{z}||_1 : A\mathbf{z} = A\mathbf{x}\}$.

Compressive Sensing

Null Space Conditions: ℓ_0

- $N(A)$ is the null space of the measurement matrix A.

Lemma

Matrix $A \in \mathbb{R}^{m \times n}$ has (ℓ_0, t)-recoverability if and only if $N(A) \setminus \{0\}$ contains no $(2t)$-sparse vector.

Compressive Sensing

Null Space Conditions: ℓ_1

- When C is a set of coordinate indices of a vector $\mathbf{y}$, $\mathbf{y}_{|C}$ is the vector restricted to the indices in C.

Lemma

Matrix $A \in \mathbb{R}^{m \times n}$ has (ℓ_1, t)-recoverability if and only if for every $y \in N(A) \setminus \{0\}$ and every $C \subset \{1, \ldots, n\}$ with $|C| = t$, $||\mathbf{y}_{|C}||_1 < \frac{1}{2}||\mathbf{y}||_1$.

Column Replacement

- The *column replacement* technique:
  - Given an $N \times k$ matrix A with columns indexed by $\{1, \ldots, k\}$, and an $M \times \ell$ *pattern matrix* P with *symbols* from $\{1, \ldots, k\}$
  - For each entry of P, which is necessarily in $\{1, \ldots, k\}$, select the corresponding column of A
  - The result is an $MN \times \ell$ matrix having entries chosen from the set of entries of A

Column Replacement

Example: *B* is the column replacement of *A* into *P*

- P: 2×4 pattern matrix with *symbols* from $\{1, \ldots, 3\}$
- A: 2×3 matrix with columns indexed by $\{1, \ldots, 3\}$

$$P = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} a & b & a \\ b & a & a \end{bmatrix}$$

$$B = \left[\begin{array}{cccc} a & b & a & a \\ b & a & a & b \\ \hline a & a & b & a \\ a & b & a & b \end{array}\right]$$

- B: 4×4 matrix having entries chosen from the set of entries of A

Column Replacement and Hash Families

- Our goal is to ensure that when A meets one of the null space conditions for sparsity t, B does as well
- Not every pattern matrix will do
- What are the requirements for a pattern matrix such that the sparsity supported by B is at least that of A?

Perfect Hash Families

Definition and Example

- A *perfect hash family* PHF$(N; k, v, t)$ is an $N \times k$ array on v symbols, in which in every $N \times t$ subarray, at least one row consists of distinct symbols
- Example: a PHF$(6; 12, 3, 3)$

$$\begin{bmatrix} & & & & & & & \downarrow & \downarrow & \downarrow & & \\ 0 & 1 & 2 & 2 & 1 & 2 & 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 1 & 1 & 2 & 1 \\ 2 & 0 & 2 & 1 & 2 & 1 & 0 & 2 & 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 2 & 1 & 1 & 2 & 2 & 0 & 1 & 2 & 1 \end{bmatrix}$$

- A PHF separates t columns into t parts
- We need a weaker condition

Separating Hash Families

Definition and Example

- A *separating hash family* SHF$(N; k, v, \{w_1, \ldots, w_s\})$ with $t = \sum_{i=1}^{s} w_i$ is an $N \times k$ array on v symbols, in which in every $N \times t$ subarray, and every way to partition the t columns into classes of sizes $w_1, \ldots, w_s$, there is at least one row in which symbols in different classes are different
- Example: an SHF$(3; 16, 4, \{1, 2\})$
  - For the specific separation $\{11, 16\}$ from $\{15\}$

$$\begin{bmatrix} & & & & & & & & & & \downarrow & & & & \downarrow & \downarrow \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 \end{bmatrix}$$

Distributing Hash Families

Definition and Example

- A *distributing hash family* DHF$(N; k, v, t, s)$ is an SHF$(N; k, v, \{w_1, \ldots, w_s\})$ for *every* way to choose $w_1 + \cdots + w_s = t$
- Example: a DHF$(10; 13, 9, 5, 2)$; $* =$ don't care

$$\begin{bmatrix} 6 & 7 & 8 & 3 & 4 & 0 & 2 & 2 & 3 & 0 & 5 & 1 & 1 \\ 3 & 1 & 1 & 7 & 2 & 6 & 8 & 4 & 3 & 0 & 2 & 0 & 5 \\ 8 & 5 & 1 & 4 & 2 & 3 & 2 & 6 & 7 & 0 & 1 & 3 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 & 2 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 0 & 4 & 2 & 0 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 & 1 & 0 & 0 & 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & * & * & * & * & 1 & * & * & 1 & * & * & 0 & 1 \end{bmatrix}$$

Column Replacement and Recoverability

(ℓ_0, t)-Recoverability

Theorem

Suppose that

- *A is an $r \times k$ measurement matrix that meets the (ℓ_0, t)-null space condition,*
- *P is an SHF$(m; n, k, \{1, t\})$, and*
- *B is the column replacement of A into P.*

Then B is an $rm \times n$ measurement matrix that meets the (ℓ_0, t)-null space condition.

Column Replacement and Recoverability

(ℓ_1, t)-Recoverability

Theorem

Suppose that

- *A is an* $r \times k$ *measurement matrix that meets the* (ℓ_1, t)*-null space condition,*
- *P is a* $\mathrm{DHF}(m; n, k, t+1, 2)$*, and*
- *B is the column replacement of A into P.*

Then B is an $rm \times n$ *measurement matrix that meets the* (ℓ_1, t)*-null space condition.*

Extensions to Column Replacement

- Let's revisit the pattern matrix $\mathrm{DHF}(10; 13, 9, 5, 2)$:

$$\begin{bmatrix} 6 & 7 & 8 & 3 & 4 & 0 & 2 & 2 & 3 & 0 & 5 & 1 & 1 \\ 3 & 1 & 1 & 7 & 2 & 6 & 8 & 4 & 3 & 0 & 2 & 0 & 5 \\ 8 & 5 & 1 & 4 & 2 & 3 & 2 & 6 & 7 & 0 & 1 & 3 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 & 2 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 0 & 4 & 2 & 0 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 & 1 & 0 & 0 & 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & * & * & * & * & 1 & * & * & 1 & * & * & 0 & 1 \end{bmatrix}$$

- Number of symbols per row need not be the same
  - *As many A matrices as there are rows of P!*
  - *The strength of each A matrix may be different!*

An Extension to Column Replacement

$$\left[\begin{array}{ccccccccccccc}
6 & 7 & 8 & 3 & 4 & 0 & 2 & 2 & 3 & 0 & 5 & 1 & 1 \\
3 & 1 & 1 & 7 & 2 & 6 & 8 & 4 & 3 & 0 & 2 & 0 & 5 \\
8 & 5 & 1 & 4 & 2 & 3 & 2 & 6 & 7 & 0 & 1 & 3 & 0 \\
\hline
0 & 2 & 0 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\
0 & 0 & 2 & 1 & 1 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 1 \\
1 & 1 & 2 & 2 & 2 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\
1 & 0 & 1 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 2 & 2 & 1 \\
1 & 1 & 0 & 1 & 0 & 4 & 2 & 0 & 2 & 0 & 1 & 0 & 2 \\
0 & 0 & 3 & 0 & 1 & 0 & 0 & 2 & 4 & 0 & 0 & 1 & 0 \\
0 & * & * & * & * & 1 & * & * & 1 & * & * & 0 & 1
\end{array}\right]$$

- Rows 1–3 use ≤ 9 symbols; ≤ 5 symbols to separate
- Rows 4–10 use ≤ 5 symbols; ≤ 3 to separate
- *This gives great flexibility in column replacement*
    - ...and, also in recovery

Column Replacement Revisited

- The *column replacement* technique:
  - Given an $M \times \ell$ *pattern matrix* P with *symbols* in row ρ from $\{1, \ldots, k_\rho\}$ for $1 \leq \rho \leq M$, and
  - for each $1 \leq \rho \leq M$, an $N_\rho \times k_\rho$ matrix A with columns indexed by $\{1, \ldots, k_\rho\}$
  - For each entry in row ρ of P, which is necessarily in $\{1, \ldots, k_\rho\}$, select the corresponding column of A_ρ
  - The result is an $(\sum_{\rho=1}^{M} N_\rho) \times \ell$ matrix having entries chosen from the set of entries of the $\{A_\rho\}$
- The theorems for recoverability extend in an 'obvious' way, but ...
- How do we use different strength ingredients?

Strengthening Hash Families

- In row ρ, we say a separation is *effective* only if on the columns used in the separation, the number of symbols used does not exceed m_ρ
- and every row ρ may have a different threshold —
- while $m_\rho \leq t + 1$, it is very possible that $m_\rho < t + 1$

- if this holds, then for row ρ we need only a $(N_\rho \times k_\rho)$ measurement matrix for sparsity $m_\rho - 1$, and hence can use ingredient matrices that support lower sparsity!

Preliminary Results

- Trade-offs: may be able to use fewer A_ρ if they have higher strength
  - Sometimes this seems to save a lot, sometimes not!
- We developed a simple greedy algorithm to find a pattern matrix based on the Stein-Lovász method
- The colours correspond to different kinds of hash families

Preliminary Results from Heuristic Search

DHF(N; 13, v, t, s)

Number of symbols v (up to 9 shown)

s	m	t	2	3	4	5	6	7	8	9
2	2	2	4	3	2	2	2	2	2	2
	2	3	10	9	10	10	10	10	10	10
	2	4	22	23	22	23	22	21	21	22
	2	5	58	60	60	61	61	59	61	61
	2	6	127	126	131	126	130	132	126	127
	3	3		6	4	3	3	3	3	3
	3	4		11	11	11	11	11	11	11
	3	5		27	25	26	26	27	26	26
	3	6		58	61	61	60	59	60	59
	4	4			7	6	5	4	4	4
	4	5			15	14	15	14	14	14
	4	6			32	33	33	31	33	33
	5	5				10	8	6	6	5
	5	6				20	16	15	17	16
	6	6					14	10	8	6

Results after a Simple Post-Optimization

DHF(N; 13, v, t, s)

Number of symbols v (up to 7 shown)

s	m	t	2	3	4	5	6	7
2	2	2	4	3	2	2	2	2
	2	3	10/9					
	2	4	22/19					
	2	5	58/50					
	2	6	127/106					
	3	3		6/5	4	3	3	3
	3	4		11/10				
	3	5		27/24				
	3	6		58/53				
	4	4			7/–	6/5	5	4
	4	5			15/14			
	4	6			32/30			
	5	5				10/9	8/7	6
	5	6				20/18	16/15	
	6	6					14/13	10/–

Conclusions

- The observation that each row of the pattern matrix could use a different measurement matrix gives great flexibility in the construction
  - B is the column replacement of A_ρ into row ρ of P, $1 \leq \rho \leq M$
- Furthermore, the strength of each A_ρ need not be the same
- Recovery is also affected
  - Indeed, the recovery technique need not be the same for each of the A_ρ matrices
- This flexibility in construction and recovery deserves more investigation

Current Research

1. Develop construction techniques for pattern matrices (hash families)
2. Investigate the trade-offs between hash family size and the sizes and strengths of the measurement matrices A_ρ
3. Investigate the trade-offs for signal recovery; now, we don't need to use the same recovery technique for each matrix!
4. Cope with noise in the signal; this requires additional conditions, both on the hash family used and the ingredient measurement matrices A_ρ