# On difference Matrices with respect to cosets

(joint work with C. Suetake)

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## §1 Difference matrices

**Definition 1.** Let $U$ be a group of order $u$ and $k, \lambda \in \mathbb{N}$. A $k \times u\lambda$ matrix $H = [d_{ij}]$ is a $(u, k\,; \lambda)$-difference matrix over $U$ if $d_{ij} \in U$ $(\forall i, j,\ 1 \leq i \leq k,\ 1 \leq j \leq u\lambda)$ and satisfies the following :

$$\sum_{1 \leq j \leq u\lambda} d_{ij} d_{\ell j}^{-1} = \lambda U \in \mathbb{Z}[U] \quad (1 \leq i \neq \ell \leq k).$$

A $(u, u\lambda\,; \lambda)$-difference matrix is called a GH$(u\lambda, U)$ matrix and is also called a GH$(u, \lambda)$ matrix over $U$.

**Example 2.** A $(3, 6\,; 2)$-difference matrix (i.e. GH$(3, 2)$ matrix) over the additive group of $GF(3)$ :

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 2 & 1 & 2 & 0 & 1 \end{bmatrix} \begin{matrix} \text{the 0th row} \\ \text{the 1st row} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$$

For example, the 1st row minus the 2nd row equals

$$(-1, -2, 1, 0, 2, 0) \equiv (2, 1, 1, 0, 2, 0) \pmod 3)$$

**Example 3**. A $(4, 8\,; 6)$-difference matrix over the additive group of $GF(4)(= K = \{0, 1, a, b\})$ :

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1&1&1&a&a&a&a&a&a&b&b&b&b&b&b \\ 0&1&a&0&1&a&1&b&b&1&b&b&1&a&b&1&a&b&0&0&a&0&0&a \\ 0&1&a&0&1&a&0&a&a&0&a&a&b&0&1&b&0&1&b&b&1&b&b&1 \\ 0&0&1&a&a&b&1&a&a&b&0&1&b&b&0&1&a&b&0&1&1&a&b&0 \\ 0&0&1&a&a&b&0&b&b&a&1&0&1&1&a&b&0&1&b&a&a&1&0&b \\ 0&1&b&a&b&1&0&1&1&a&b&a&a&1&b&0&0&0&0&1&b&a&b&a \\ 0&1&b&a&b&1&1&0&0&b&a&b&0&b&1&a&a&a&b&a&0&1&0&1 \end{bmatrix}$$

We can regard each row $h_i$ $(0 \le i \le 7)$ as an element of the group $K^{24} = K \oplus \cdots \oplus K$. We note that $W := \{h_0, h_1, h_2, h_3\}$ is a subgroup of $K^{24}$ and $h_4 + W = \{h_4, h_5, h_6, h_7\}$ is a left coset of $W$ in $K^{24}$.

## §2 Difference matrices with respect to cosets

**Definition 4.** Let $R$ be the set of rows of a $(u, k\,; \lambda)$-difference matrix $H$ over a group $U$. Then $R$ is a subset of $U^n (= U \times \cdots \times U)$, where $n = u\lambda$. Let $W$ be a subset of $R$. We say $H$ is of coset type with respect to $W$ if the following conditions are satisfied :

(i) $W$ is a subgroup of $U^n$.

(ii) If $w \in W$ and $r \in R$, then $rw \in R$.

- Clearly $R$ is a union of some right cosets $gW$ $(g \in U^n)$.
- $H$ is said to be of coset type with respect to a row $w$ of $H$ if $H$ is of coset type w.r.t. $\langle w \rangle$. We note that $w^0, w, w^2, \cdots \in R$.

In particular, $w^0 = (1, \cdots, 1) \in R$ and $\exp(U)$ is a prime.

- W. de Launey considered the case that $R = W$ and $k = u\lambda$ and called $H$ a group Hadamard matrix (1983).
- T.P. McDonough, V.C. Mavron and C.A. Pallikaros studied GH$(u\lambda, U)$ matrices $H$ of coset type with respect to some row of $H$ and showed that $U$ is an elementary abelian $p$-group (2000).

# Trivial difference matrices of coset type



If we permutes the rows or columns, multiplies, on the left, the entries of any row or, on the right, the entries of any column by any group element, we also obtain a difference matrix, which we call "equivalent".

Let $p$ be a prime and let $U = \{g_1, \cdots, g_q\}, g_1 = 1$ be any $p$-group of exponent $p$. Then a $(q, p; \lambda)$-difference matrix $H$ of coset type with respect to a row is equivalent to the following:

$$H = \begin{bmatrix} w^0 \\ w^1 \\ \vdots \\ w^{p-1} \end{bmatrix},$$

where $w = (Jg_1, Jg_2, \cdots, Jg_q), \; J = (1, \cdots, 1)$ ($\lambda$ times).

We say that $H$ is a trivial difference matrix of coset type. We note that $\lambda$ is arbitrary.

**Question.** What can we say about the parameter $\lambda$ and the structure of nontrivial difference matrices of coset type ?

# A $(q, 2p; q)$-difference matrix of coset type

**Example 5**. Let $U = \{g_1(=1), \cdots, g_q\}$ be a $p$-group of order $q$ and exponent $p$ with $p$ an odd prime. Set $g = (g_1, \cdots, g_q) \in U^q$ and $J = (1, \cdots, 1) \in U^q$. We can verify that the following matrix $H$ is a $(q, 2p; q)$-difference matrix over $U$.

$$H = \begin{bmatrix} W \\ hW \end{bmatrix}, \quad \text{where } W = \begin{bmatrix} w^0 \\ w^1 \\ \vdots \\ w^{p-1} \end{bmatrix} \text{ and}$$

$$w = (Jg_1, Jg_2, \cdots, Jg_q), \; h = (g, g, \cdots, g) \; (q \text{ times})$$

Therefore, $H$ is of coset type with respect to the row $w$.
This example shows that the $p$-group corresponding to $(q, k, \lambda)$-difference matrix of coset type is not always abelian even if $k > p$.

# A $(3, 18\,; 6)$-difference matrix

**Example 6.** The following matrix $H$ is a $(3, 18\,; 6)$-difference matrix (i.e. $\mathrm{GH}(18, \mathbb{Z}_3)$) w.r.t. $W = \langle w \rangle$, where $w$ is the 7 th row of $H$. We note that the $i$ th, $(i + 6)$ th and $(i + 12)$ th rows constitute a right coset of $\langle w \rangle$ in $U^{18}$ for $i \in \{0, 1, \cdots, 5\}$, where $U = \langle a \rangle \simeq \mathbb{Z}_3$.

$$
H = \left[\begin{array}{cccccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & a & a^2 & a & a^2 & 1 & a^2 & a^2 & a & a & 1 & 1 & a & 1 & a & a^2 & a^2 \\
1 & a^2 & a^2 & a & a & 1 & 1 & a & 1 & a & a^2 & a^2 & 1 & 1 & a & a^2 & a & a^2 \\
1 & a & 1 & a & a^2 & a^2 & 1 & 1 & a & a^2 & a & a^2 & 1 & a^2 & a^2 & a & a & 1 \\
1 & a & a^2 & a^2 & 1 & a & 1 & a & a^2 & a^2 & 1 & a & a & a^2 & 1 & 1 & a & a^2 \\
1 & a^2 & a & 1 & a^2 & a & 1 & a^2 & a & 1 & a^2 & a & 1 & a^2 & a & 1 & a^2 & a \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & a & a & a & a & a & a & a^2 & a^2 & a^2 & a^2 & a^2 & a^2 \\
1 & 1 & a & a^2 & a & a^2 & a & 1 & 1 & a^2 & a^2 & a & a^2 & 1 & a^2 & 1 & a & a \\
1 & a^2 & a^2 & a & a & 1 & a & a^2 & a & a^2 & 1 & 1 & a^2 & a^2 & 1 & a & 1 & a \\
1 & a & 1 & a & a^2 & a^2 & a & a & a^2 & 1 & a^2 & 1 & a^2 & a & a & 1 & 1 & a^2 \\
1 & a & a^2 & a^2 & 1 & a & a & a^2 & 1 & 1 & a & a^2 & 1 & a & a^2 & a^2 & 1 & a \\
1 & a^2 & a & 1 & a^2 & a & a & 1 & a^2 & a & 1 & a^2 & a^2 & a & 1 & a^2 & a & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & a^2 & a^2 & a^2 & a^2 & a^2 & a^2 & a & a & a & a & a & a \\
1 & 1 & a & a^2 & a & a^2 & a^2 & a & a & 1 & 1 & a^2 & a & a^2 & a & a^2 & 1 & 1 \\
1 & a^2 & a^2 & a & a & 1 & a^2 & 1 & a^2 & 1 & a & a & a & a & a^2 & 1 & a^2 & 1 \\
1 & a & 1 & a & a^2 & a^2 & a^2 & a^2 & 1 & a & 1 & a & a & 1 & 1 & a^2 & a^2 & a \\
1 & a & a^2 & a^2 & 1 & a & a^2 & 1 & a & a & a^2 & 1 & a^2 & 1 & a & a & a^2 & 1 \\
1 & a^2 & a & 1 & a^2 & a & a^2 & a & 1 & a^2 & a & 1 & a & 1 & a^2 & a & 1 & a^2
\end{array}\right]
\begin{array}{l}
\leftarrow w^0 \\ \\ \\ \\ \\ \\
\leftarrow w \\ \\ \\ \\ \\ \\
\leftarrow w^2 \\ \\ \\ \\ \\ \\
\end{array}
$$

# §3 $\mathrm{TD}_\lambda(k,u)$'s obtained from difference matrices

A *transversal design* $\mathrm{TD}_\lambda(k,u)$ $(u > 1)$ is an incidence structure $\mathcal{D} = (\mathbb{P}, \mathbb{B})$, where

(i) $\mathbb{P}$ is a set of $ku$ points partitioned into $k$ classes $C_1, \cdots, C_k$ (called *point classes*), each of size $u$,

(ii) $\mathbb{B}$ is a collection of $k$-subsets of $\mathbb{P}$ (called blocks),

(iii) Any two distinct points in the same point class are incident with no block and any two points in distinct point classes are incident with exactly $\lambda$ blocks.

# A $\mathrm{TD}_\lambda(k, u)$ obtained from a difference matrix

**Definition 7.** Let $H = [h_{ij}]$ be a $(u, k\,; \lambda)$-difference matrix over a group $U$ of order $u$ and set $n = u\lambda$. Let $h_i = (h_{i1}, \cdots, h_{in})(\in U^n)$ be the $i$ th row of $H$.

An incidence structure $\mathcal{D}_H(\mathbb{P}, \mathbb{B})$ obtained from $H$ is defined by

the set of points : $\mathbb{P} = \{(i, x) \mid 0 \leq i \leq k-1,\ x \in U\},\ |\mathbb{P}| = ku$

the set of blocks : $\mathbb{B} = \{B_{j,y} \mid 1 \leq j \leq n,\ y \in U\}$, where

$$B_{j,y} = \{(1, h_{1j}y), (2, h_{2j}y), \cdots, (n, h_{kj}y)\}$$

incidence : $(i, a) \in B_{j,b}$ $(\longleftrightarrow\ a = h_{ij}b)$.

We note that each block is defined by using a column of $H$ or its translate.

The following is well known.

- $\mathcal{D}_H(\mathbb{P}, \mathbb{B})$ is a $\mathrm{TD}_\lambda(k, u)$ :

$$C_i = \{i\} \times U \ \ (0 \leq i \leq k-1) \ \ \text{(the point classes)}$$

$$\mathcal{B}_j = \{B_{j,y} \mid y \in U\} \ \ (1 \leq j \leq n) \ \ \text{(parallel classes)}$$

- The action of $U$ on $(\mathbb{P}, \mathbb{B})$ is defined by

$(i, a)^{\rho(x)} = (i, ax),\ \ B_{j,b}{}^{\rho(x)} = B_{j,bx}$. Then, $\rho(U) \leq \mathrm{Aut}(\mathbb{P}, \mathbb{B})$ and acts regularly on each $C_i$ and $\mathcal{B}_j$.

If $H$ is of coset type, we obtain another automorphism.

**Lemma 8**. Assume a $(u,k;\lambda)$-difference matrix $H$ is of coset type w.r.t a row $h_m$ of $H$. The action $\theta(h_m)$ on $\mathcal{D}_H(\mathbb{P},\mathbb{B})$ is defined by

$$(i,a)^{\theta(h_m)} = (\ell,a), \text{ where } h_\ell = h_i h_m \text{ and } H = \begin{bmatrix} h_0 \\ \vdots \\ h_{k-1} \end{bmatrix}.$$

$$B_{j,b}{}^{\theta(h_m)} = B_{j,{h_{mj}}^{-1}b}$$

Then the following holds.

(i) $\theta(h_m) \in \mathrm{Aut}(\mathbb{P},\mathbb{B})$ and $\theta(h_m)$ leaves each parallel class $\mathcal{B}_j$ $(= \{B_{j,y} \mid y \in U\})$ invariant and acts semiregularly on $\mathbb{P}$.

(ii) $\theta(h_m)$ fixes each block of $\mathcal{B}_j$ if $h_{mj} = 1$, and no block of $\mathcal{B}_j$ otherwise.

(iii) $[\rho(U), \langle\theta(h_m)\rangle] = 1$ and $\rho(U) \times \langle\theta(h_m)\rangle$ is semi-regular on $\mathbb{P}$.

Concerning the parameter $\lambda$ of $(u, u\lambda; \lambda)$-difference matrix, we can prove the following as an application of Lemma 8.

**Proposition 9**. Assume $H$ is a $(u, u\lambda; \lambda)$-difference matrix over a group $U$ (i.e. GH$(u\lambda, U)$. If $H$ is of coset type with respect to a row of $H$, then either $\lambda = 1$ or $\exp(U) \mid \lambda$.

We now consider the case that $U \simeq \mathbb{Z}_p$.

**Notation 10.** Let $p$ be a prime and let $U = \langle a \rangle \simeq \mathbb{Z}_p$ be a group of order $p$. Set

$$N = U^{\lambda} = U \times \cdots \times U, \quad G = N^p \,(= N \times \cdots \times N \simeq U^{\lambda p})$$

and identify $G$ with $U^{p\lambda}$. Set $J = (1, \cdots, 1) \in U^{\lambda}$ and $w = (J, Ja, \cdots, Ja^{p-1}) \in G$, where $(x_1, \cdots, x_\lambda)x = (x_1 x, \cdots, x_\lambda x)$ for $(x_1, \cdots, x_\lambda) \in N$ and $x \in U$.

For $z = (z_1, \cdots, z_m) \in U^m$, we set $\widehat{z} = z_1 + \cdots + z_m \in \mathbb{Z}[U]$.

**Remark 11.** Let $H$ be a $(p, k\,; \lambda)$-difference matrix over $U$. If $H$ is of coset type with respect to a row $w$ of $H$, then by permuting columns of $H$ if necessary, we may assume that $w = (J, Ja, \cdots, Ja^{p-1})$.

Throughout the rest of this section we assume the hollowing.

**Hypothesis 12.**

(i) $H$ is a $(p, k\,; \lambda)$-difference matrix over $U \simeq \mathbb{Z}_p$ with $p$ a prime.

(ii) $H$ is of coset type with respect to a row $w$ of $H$ and nontrivial.

Since $H$ is of coset type, we have $k = pr$ for an integer $r$.
From now on we use the notation

$$w = (J, Ja, \cdots, Ja^{p-1}).$$

According to the form of $w$, we write each row $v$ of $H$ in the form

$$v = (v_0, v_1, \cdots, v_{p-1}) \in (U^\lambda)^p, \quad \text{where } v_i \in N = U^\lambda.$$

We call $v_i$ the $i$th part of $v$.
By permuting rows of $H$ if necessary, we may assume that a $(p, k\,; \lambda)$-difference matrix $H$ of coset type over $U$ has the following form.

$$H = [M, w] := \begin{bmatrix} M \\ Mw \\ \vdots \\ Mw^{p-1} \end{bmatrix}, \quad \text{where } M = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{r-1} \end{bmatrix} \qquad (1)$$

is an $r \times p\lambda$ matrix over $U$ $(r > 1)$ and
$h_0 := (J, \cdots, J) \in U^{p\lambda}$, $w = (J, Ja, \cdots, Ja^{p-1}) \in U^{p\lambda}$.
We note that $W := \langle w \rangle \simeq \mathbb{Z}_p$ and $W \subset R$(= the set of rows of $H$).
and $R$ is a union of $r$ cosets of $W$ in $U^{p\lambda}$. Moreover, $\{h_0, \cdots, h_{r-1}\}$ is the representatives of the $r$ cosets.

We say $H(= [M, w])$ in (1) is a **standard** $(p, k\,;\lambda)$-difference matrix of coset type w.r.t. $w$ over $U$.

- The $(ir + j)$ th row of $H$ is $h_j w^i$.
- $\widehat{h_j w^i} = \lambda U \;\; \forall (i, j) \neq (0, 0) \quad \because h_0 \in R$

Fix $v = h_j w^i$ and set $v = (v_0, \cdots, v_{p-1}),\ v_i \in U^\lambda$ and $\widehat{v_i} = m_{i,0}1 + m_{i,1}a + m_{i,2}a^2 + \cdots + m_{i,p-1}a^{p-1}$, where $m_{ij}$'s are non-negative integers. Considering submatrices $[[h_0, v, vw, \cdots, vw^{p-1}]]$ of $H$. We have $p^2 + p$ linear equations:
$\sum_{0 \le j \le p-1} m_{ij} = \lambda \quad (0 \le i \le p-1)$
$m_{0,s} + m_{1,s-t} + m_{2,s-2t} + \cdots + m_{0,s-(p-1)t} = \lambda\ (0 \le \forall s, t \le p-1)$.

The set of simultaneous linear equations over $\mathbb{Z}$ in $p^2$ indeterminates $m_{ij}$ gives

Lemma. $p \mid \lambda$ and $\widehat{v_i} = (\lambda/p)U\ (0 \le i \le p-1)$.

**Theorem 13.** Let $p$ be a prime and let $H$ be a nontrivial $(p, k\,; \lambda)$-difference matrix of coset type w. r. t. a row over $U$. Then $p \mid \lambda$ and using some normalized $(p, k/p\,; \lambda/p)$-difference matrices $H_0, H_1, \cdots, H_{p-1}$ over $U$, $H$ is equivalent to the following standard form.

$$\begin{bmatrix} (H_0, H_1, \cdots, H_{p-1}) \\ (H_0, H_1, \cdots, H_{p-1})w \\ \vdots \\ (H_0, H_1, \cdots, H_{p-1})w^{p-1} \end{bmatrix}, \qquad (2)$$

where $w = (J, Ja, \cdots, Ja^{p-1}) \in U^{p\lambda}$, $J = (1, \cdots, 1) \in U^{\lambda}$.

Conversely, given any normalized $(p, k/p\,; \lambda/p)$-difference matrices $H_0, H_1, \cdots, H_{p-1}$ over $U$, the matrix given by (2) is a $(p, k\,; \lambda)$-difference matrix over $U$ w.r.t. $w$.

In Theorem 13, set $k = u\lambda$. Then we have

**Corollary 14**. If $H$ is a $\text{GH}(p\lambda, \mathbb{Z}_p)$ matrix of coset type, then there exist normalized $\text{GH}(\lambda, \mathbb{Z}_p)$ matrices $H_i$'s $(0 \leq i \leq p-1)$ such that $H$ is equivalent to the following standard form.

$$\begin{bmatrix} (H_0, H_1, \cdots, H_{p-1}) \\ (H_0, H_1, \cdots, H_{p-1})w \\ \vdots \\ (H_0, H_1, \cdots, H_{p-1})w^{p-1} \end{bmatrix}, \quad \text{where } w = (J, Ja, \cdots, Ja^{p-1})$$

**Example 15**. The following is a $\mathrm{GH}(\mathbf{18}, \boldsymbol{U})$ matrix of coset type w.r.t. $\boldsymbol{w}$, where $\boldsymbol{U} = \langle \boldsymbol{a} \rangle \simeq \mathbb{Z}_3$ and $\boldsymbol{\lambda} = \mathbf{6}$.

$$
\left[\begin{array}{cccccc|cccccc|cccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&a&a^2&a&a^2&1&a^2&a^2&a&a&1&1&a&1&a&a^2&a^2\\
1&a^2&a^2&a&a&1&1&a&1&a&a^2&a^2&1&1&a&a^2&a&a^2\\
1&a&1&a&a^2&a^2&1&1&a&a^2&a&a^2&1&a^2&a^2&a&a&1\\
1&a&a^2&a^2&1&a&1&a&a^2&a^2&1&a&a&a^2&1&1&a&a^2\\
1&a^2&a&1&a^2&a&1&a^2&a&1&a^2&a&1&a^2&a&1&a^2&a\\
\hline
1&1&1&1&1&1&a&a&a&a&a&a&a^2&a^2&a^2&a^2&a^2&a^2\\
1&1&a&a^2&a&a^2&a&1&1&a^2&a^2&a&a^2&1&a^2&1&a&a\\
1&a^2&a^2&a&a&1&a&a^2&a&a^2&1&1&a^2&a^2&1&a&1&a\\
1&a&1&a&a^2&a^2&a&a&a^2&1&a^2&1&a^2&a&a&1&1&a^2\\
1&a&a^2&a^2&1&a&a&a^2&1&1&a&a^2&1&a&a^2&a^2&1&a\\
1&a^2&a&1&a^2&a&a&1&a^2&a&1&a^2&a^2&a&1&a^2&a&1\\
\hline
1&1&1&1&1&1&a^2&a^2&a^2&a^2&a^2&a^2&a&a&a&a&a&a\\
1&1&a&a^2&a&a^2&a^2&a&a&1&1&a^2&a&a^2&a&a^2&1&1\\
1&a^2&a^2&a&a&1&a^2&1&a^2&1&a&a&a&a&a^2&1&a^2&1\\
1&a&1&a&a^2&a^2&a^2&a^2&1&a&1&a&a&1&1&a^2&a^2&a\\
1&a&a^2&a^2&1&a&a^2&1&a&a&a^2&1&a^2&1&a&a&a^2&1\\
1&a^2&a&1&a^2&a&a^2&a&1&a^2&a&1&a&1&a^2&a&1&a^2
\end{array}\right]
\begin{array}{l}
\\ \\ \\ =M \\ \\ \\
\leftarrow w \\ \\ \\ =Mw \\ \\ \\
\\ \\ \\ =Mw^2 \\ \\ \\
\end{array}
$$

The following three GH$(6, U)$ matrices are obtained form the GH$(18, U)$ as its submatrices, where $U = \langle a \rangle \simeq \mathbb{Z}_3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & a & a^2 & a & a^2 \\ 1 & a^2 & a^2 & a & a & 1 \\ 1 & a & 1 & a & a^2 & a^2 \\ 1 & a & a^2 & a^2 & 1 & a \\ 1 & a^2 & a & 1 & a^2 & a \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a^2 & a^2 & a & a & 1 \\ 1 & a & 1 & a & a^2 & a^2 \\ 1 & 1 & a & a^2 & a & a^2 \\ 1 & a & a^2 & a^2 & 1 & a \\ 1 & a^2 & a & 1 & a^2 & a \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & 1 & a & a^2 & a^2 \\ 1 & 1 & a & a^2 & a & a^2 \\ 1 & a^2 & a^2 & a & a & 1 \\ a & a^2 & 1 & 1 & a & a^2 \\ 1 & a^2 & a & 1 & a^2 & a \end{bmatrix}$$

**Example 16.** Let $H_1$ and $H_2$ be any normalized Hadamard matrices of order $n$. Then, $H = \begin{bmatrix} H_1 & H_2 \\ H_1 & -H_2 \end{bmatrix}$ is also an Hadamard matrix of order $2n$ with respect to the rows $(1, \cdots, 1, -1, \cdots, -1)$ of $H$

By a similar argument as in the proof of Theorem 13 we have the following.

**Proposition 17**. Let $\alpha$ be a primitive element of GF$(q)$ Let $J = (1, \cdots, 1)$ of length $q\mu$. Set $w = (0J, 1J, \alpha J, \cdots, \alpha^{q-2}J)$. Let $H_0, H_1, \cdots, H_{q-1}$ be any normalized GH$(q\mu, U)$ matrices, where $U = GF(q)^+$. Let $M = (H_0, H_1, \cdots, H_{q-1})$ be a $q\mu \times q^2\mu$ matrix over $U$. Then the following matrix $H$ is a GH$(q^2\mu, U)$ matrix of coset type w.r.t $w$.

$$H = \begin{bmatrix} M \\ M + w \\ M + w\alpha \\ \vdots \\ M + w\alpha^{q-2} \end{bmatrix}$$

# A GH($48, \mathbb{Z}_2 \times \mathbb{Z}_2$) matrix

**Example 18.** The following are GH$(12, GF(4)^+)$ matrices.

$$H_1 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&1&1&a&a&a&a^2&a^2&a^2 \\ 0&0&0&a&a&a&a^2&a^2&a^2&1&1&1 \\ 0&1&a&1&a&a^2&0&1&a^2&0&a&a^2 \\ 0&1&a&a&a^2&1&1&a^2&0&a&a^2&0 \\ 0&1&a&a^2&1&a&a^2&0&1&a^2&0&a \\ 0&a&a^2&0&1&a^2&1&a&a^2&a&0&1 \\ 0&a&a^2&1&a^2&0&a&a^2&1&0&1&a \\ 0&a&a^2&a^2&0&1&a^2&1&a&1&a&0 \\ 0&a^2&1&0&a^2&a&a&1&0&a^2&a&1 \\ 0&a^2&1&a&0&a^2&0&a&1&1&a^2&a \\ 0&a^2&1&a^2&a&0&1&0&a&a&1&a^2 \end{bmatrix} \quad H_2 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&1&1&a&a&a&a^2&a^2&a^2 \\ 0&0&0&a&a&a&a^2&a^2&a^2&1&1&1 \\ 0&1&a&1&a&a^2&0&1&a^2&0&a&a^2 \\ 0&1&a&a&a^2&1&1&a^2&0&a&a^2&0 \\ 0&1&a&a^2&1&a&a^2&0&1&a^2&0&a \\ 0&a&a^2&0&1&a^2&a&a^2&1&1&a&0 \\ 0&a&a^2&1&a^2&0&a^2&1&a&a&0&1 \\ 0&a&a^2&a^2&0&1&1&a&a^2&0&1&a \\ 0&a^2&1&0&a^2&a&0&a&1&a&1&a^2 \\ 0&a^2&1&a&0&a^2&1&0&a&a^2&a&1 \\ 0&a^2&1&a^2&a&0&a&1&0&1&a^2&a \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&1&1&a&a&a&a^2&a^2&a^2 \\ 0&0&0&a&a&a&a^2&a^2&a^2&1&1&1 \\ 0&1&a&1&a&a^2&0&1&a^2&0&a&a^2 \\ 0&1&a&a&a^2&1&1&a^2&0&a&a^2&0 \\ 0&1&a^2&a&0&a^2&a&0&1&a^2&1&a \\ 0&a&1&a^2&1&a&0&a^2&1&0&a^2&a \\ 0&a&a^2&0&1&a^2&1&a&a^2&a&0&1 \\ 0&a&a^2&a^2&0&1&a^2&1&a&1&a&0 \\ 0&a^2&1&0&a^2&a&a&1&0&a^2&a&1 \\ 0&a^2&1&a^2&a&0&1&0&a&a&1&a^2 \\ 0&a^2&a&1&a^2&0&a^2&a&1&1&0&a \end{bmatrix} \quad H_4 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&1&1&a&a&a&a^2&a^2&a^2 \\ 0&0&0&a&a&a&a^2&a^2&a^2&1&1&1 \\ 0&1&a&1&a&a^2&0&1&a^2&0&a&a^2 \\ 0&1&a&a&a^2&1&1&a^2&0&a&a^2&0 \\ 0&1&a^2&a^2&0&a&a&1&0&a^2&1&a \\ 0&a&1&a^2&a&1&a^2&0&1&0&a^2&a \\ 0&a&a^2&0&1&a^2&a&a^2&1&1&a&0 \\ 0&a&a^2&1&a^2&0&a^2&1&a&a&0&1 \\ 0&a^2&1&0&a^2&a&0&a&1&a&1&a^2 \\ 0&a^2&1&a&0&a^2&1&0&a&a^2&a&1 \\ 0&a^2&a&a^2&1&0&1&a&a^2&1&0&a \end{bmatrix}$$

Then $M = (H_1, H_2, H_3, H_4)$ is a $12 \times 48$ matrix over $U = GF(4)^+$ and $H = [M, w]$ is a GH$(48, U)$ matrix of coset type w.r.t $w$.

# Question



**Question.** If $U \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ with $p$ a prime, what can we say about the structure of $(p^2, k, \lambda)$-difference matrices of coset type over $U$ ?


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[2] T.P. McDonough, V.C. Mavron and C.A. Pallikaros, Generalised Hadamard matrices and translations, Journal of Statistical Planning and Inference, Vol. 86 (2000), 527-533.

[3] T. Beth, D. Jungnickel and H. Lenz, ”Design Theory” Volume I, Second Edition, Cambridge University Press, 1999.

Thank you !