Ancestry in the face of competition, v0.1: Directed random walk on the directed percolation cluster

Matthias Birkner

Johannes-Gutenberg-Universität Mainz

Based on joint work in progress with J. Černý, A. Depperschmidt and N. Gantert

Workshop on Discrete Mathematics and Probability in Population Biology and Genetics

Institute for Mathematical Sciences
Singapore, 29th March 2011

Remark. The catchier part of the title is due to Steve Evans, who invented it in Oberwolfach in August 2005.

WIAS Leibniz Gemeinschaft

Survival in the face of competition

Matthias Birkner
Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin

joint work in progress with

Andrej Depperschmidt
TU Berlin

Workshop Mathematical Population Genetics 23 August 2005 1(30) WIAS

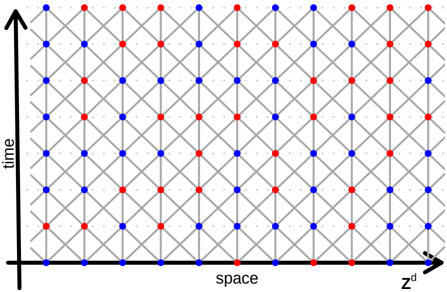
General aim:
Study/understand the space-time embedding of ancestral lineages in spatial models for populations with local density regulation (in particular, with non-constant local population sizes).

Outline

General aim:
Study/understand the space-time embedding of ancestral lineages in spatial models for populations with local density regulation (in particular, with non-constant local population sizes).

1. Directed percolation
2. Random walk on the cluster
   - A renewal structure
3. Locally regulated populations (and ancestral lineages)

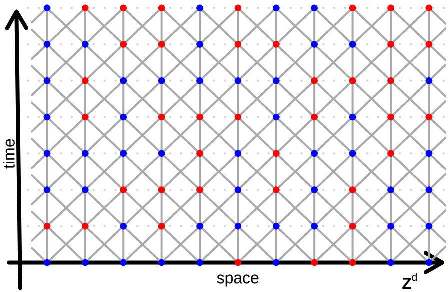
Directed (site) percolation



$p \in (0,1)$, $\omega(x,n)$, $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$ i.i.d. Bernoulli(p).

Interpretation: $\omega(x,n) = 1$: site (x,n) is open, otherwise closed

Directed (site) percolation

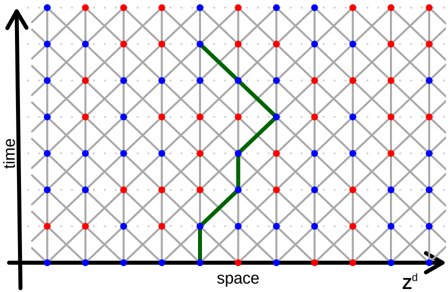


$p \in (0, 1)$, $\omega(x, n)$, $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$ i.i.d. Bernoulli(p).
Interpretation: $\omega(x, n) = 1$: site (x, n) is open, otherwise closed

Let $U \subset \mathbb{Z}^d$ be a finite, $\mathbb{Z}^d$-symmetric set with $0 \in U$

$m < n$, $x, y \in \mathbb{Z}^d$: $(x, m) \to (y, n)$ if there exist $x = x_0, x_1, \dots, x_{n-m} = y$ such that $x_i - x_{i-1} \in U$ and $\omega(x_i, m + i) = 1$ for $i = 1, \dots, n - m$ and

Directed (site) percolation

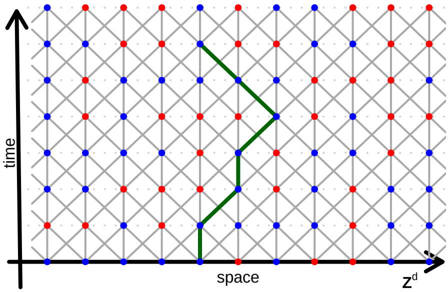


$p \in (0,1)$, $\omega(x,n)$, $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$ i.i.d. Bernoulli(p).
Interpretation: $\omega(x,n) = 1$: site (x,n) is open, otherwise closed

Let $U \subset \mathbb{Z}^d$ be a finite, $\mathbb{Z}^d$-symmetric set with $0 \in U$

$m < n$, $x, y \in \mathbb{Z}^d$: $(x,m) \to (y,n)$ if there exist $x = x_0, x_1, \ldots, x_{n-m} = y$ such that $x_i - x_{i-1} \in U$ and $\omega(x_i, m+i) = 1$ for $i = 1, \ldots, n-m$ and

Directed (site) percolation



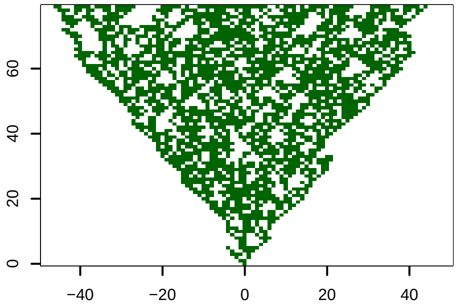
$p \in (0, 1)$, $\omega(x, n)$, $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$ i.i.d. Bernoulli(p).
Interpretation: $\omega(x, n) = 1$: site (x, n) is open, otherwise closed

Let $U \subset \mathbb{Z}^d$ be a finite, $\mathbb{Z}^d$-symmetric set with $0 \in U$

$m < n$, $x, y \in \mathbb{Z}^d$: $(x, m) \to (y, n)$ if there exist $x = x_0, x_1, \dots, x_{n-m} = y$ such that $x_i - x_{i-1} \in U$ and $\omega(x_i, m + i) = 1$ for $i = 1, \dots, n - m$ and

$\mathcal{C}_0 := \{(y, n) : y \in \mathbb{Z}^d, n \geq 0, (0, 0) \to (y, n)\}$ is the (directed) cluster of the origin

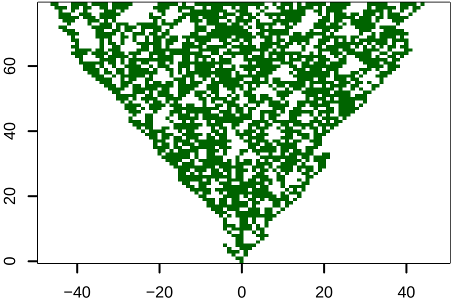
Critical value



There exists $p_c \in (0,1)$ such that

$$\mathbb{P}(|\mathcal{C}_0| = \infty) > 0 \quad \text{iff} \quad p > p_c.$$

Critical value



There exists $p_c \in (0,1)$ such that

$$\mathbb{P}(|\mathcal{C}_0| = \infty) > 0 \quad \text{iff} \quad p > p_c.$$

If $p > p_c$, $\mathbb{P}(\mathcal{C}_0 \text{ reaches height } n \mid |\mathcal{C}_0| < \infty) \le Ce^{-cn}$ for some $c, C \in (0, \infty)$.

The discrete time contact process and directed percolation

$\eta_n(x)$, $n \in \mathbb{Z}_+$, $x \in \mathbb{Z}^d$ with values in $\{0, 1\}$.
Site x is generation n is "inhabited" (or: "infected") if $\eta_n(x) = 1$.

The discrete time contact process and directed percolation

$\eta_n(x)$, $n \in \mathbb{Z}_+$, $x \in \mathbb{Z}^d$ with values in $\{0,1\}$.
Site x is generation n is "inhabited" (or: "infected") if $\eta_n(x) = 1$.

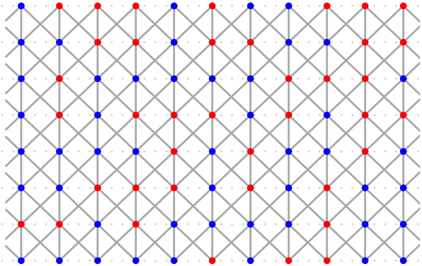
Dynamics: $U \subset \mathbb{Z}^d$ finite, symmetric, $p \in (0,1)$.
Given η_n, independently for $x \in \mathbb{Z}^d$,

$$\eta_{n+1}(x) = \begin{cases} 1 & \text{w. prob. } \; p \cdot \mathbf{1}(\eta_n(y) = 1 \text{ for some } y \in x + U) \\ 0 & \text{w. prob. } \; 1 - p \cdot \mathbf{1}(\eta_n(y) = 1 \text{ for some } y \in x + U) \end{cases}$$

The discrete time contact process and directed percolation

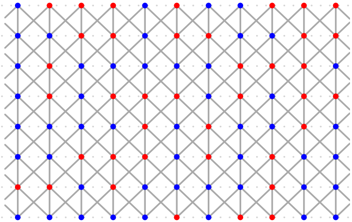
$\eta_n(x)$, $n \in \mathbb{Z}_+$, $x \in \mathbb{Z}^d$ with values in $\{0,1\}$.
Site x is generation n is "inhabited" (or: "infected") if $\eta_n(x) = 1$.
Dynamics: $U \subset \mathbb{Z}^d$ finite, symmetric, $p \in (0,1)$.
Given η_n, independently for $x \in \mathbb{Z}^d$,

$$\eta_{n+1}(x) = \begin{cases} 1 & \text{w. prob. } \; p \cdot \mathbf{1}(\eta_n(y) = 1 \text{ for some } y \in x+U) \\ 0 & \text{w. prob. } \; 1 - p \cdot \mathbf{1}(\eta_n(y) = 1 \text{ for some } y \in x+U) \end{cases}$$



$\eta_n(x) = 1$ iff $(y,0) \to (x,n)$ for some $y \in \mathbb{Z}^d$ with $\eta_0(y) = 1$.

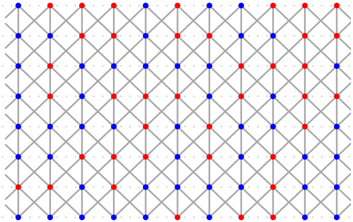
The discrete time contact process



Self duality: For $A, B \subset \mathbb{Z}^d$

$$\mathbb{P}(\eta_n(x) = 0 \text{ for all } x \in B \mid \eta_0(\cdot) = \mathbf{1}_A(\cdot))$$
$$= \mathbb{P}(\eta_n(x) = 0 \text{ for all } x \in A \mid \eta_0(\cdot) = \mathbf{1}_B(\cdot))$$

The discrete time contact process



Self duality: For $A, B \subset \mathbb{Z}^d$

$$\begin{aligned}&\mathbb{P}(\eta_n(x) = 0 \text{ for all } x \in B \mid \eta_0(\cdot) = \mathbf{1}_A(\cdot)) \\ &= \mathbb{P}(\eta_n(x) = 0 \text{ for all } x \in A \mid \eta_0(\cdot) = \mathbf{1}_B(\cdot))\end{aligned}$$

Stationary process:
For $p > p_c$, there is a (unique extremal) non-trival stationary distribution.
Informally, $\eta_0^{\mathrm{stat}}(x) = 1$ iff $\mathbb{Z}^d \times \{-\infty\} \to (x, 0)$

The discrete time contact process ...

... as a locally regulated population model

$$\eta_{n+1}(x) = \begin{cases} 1 & \text{w. prob. } \ p \cdot \mathbf{1}(\eta_n(y) = 1 \text{ for some } y \in x + U) \\ 0 & \text{w. prob. } \ 1 - p \cdot \mathbf{1}(\eta_n(y) = 1 \text{ for some } y \in x + U) \end{cases}$$

Possible interpretation for ancestry:
In generation $n+1$, each site x is inhabitable with probability p
If $\eta_n(y) = 1$ of some $y \in x + U$, the particle at y in gen. n puts an offspring at x.
If several y are eligible, one is chosen at random.

The discrete time contact process ...

... as a locally regulated population model

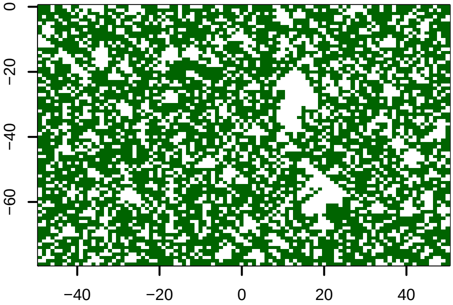
$$\eta_{n+1}(x) = \begin{cases} 1 & \text{w. prob. } \ p \cdot \mathbf{1}(\eta_n(y) = 1 \text{ for some } y \in x + U) \\ 0 & \text{w. prob. } \ 1 - p \cdot \mathbf{1}(\eta_n(y) = 1 \text{ for some } y \in x + U) \end{cases}$$

Possible interpretation for ancestry:
In generation $n + 1$, each site x is inhabitable with probability p
If $\eta_n(y) = 1$ of some $y \in x + U$, the particle at y in gen. n puts an offspring at x.
If several y are eligible, one is chosen at random.

Thus, individuals in "sparsely populated" regions have a higher reproduction probability.

An ancestral line in the discrete time contact process

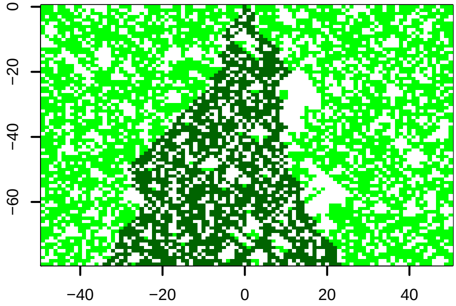
$p > p_c$, $(\eta_n^{\text{stat}}(x), x \in \mathbb{Z}^d, n \in \mathbb{Z})$ stationary DCP, assume $\eta_0^{\text{stat}}(0) = 1$.



Let X_n = position of the ancestor of the individual at the (space-time) origin n generations ago.

An ancestral line in the discrete time contact process

$p > p_c$, $(\eta_n^{\text{stat}}(x), x \in \mathbb{Z}^d, n \in \mathbb{Z})$ stationary DCP, assume $\eta_0^{\text{stat}}(0) = 1$.

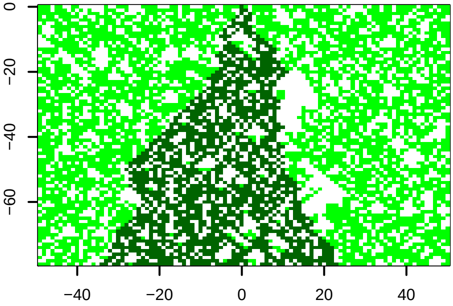


Let X_n = position of the ancestor of the individual at the (space-time) origin n generations ago. Given η^{stat} and $X_n = x$, X_{n+1} is uniform on

$$\{y \in \mathbb{Z}^d : y - x \in U, \eta_{-n-1}^{\text{stat}}(y) = 1\} \quad (\neq \emptyset).$$

An ancestral line in the discrete time contact process

$p > p_c$, $(\eta_n^{\mathrm{stat}}(x), x \in \mathbb{Z}^d, n \in \mathbb{Z})$ stationary DCP, assume $\eta_0^{\mathrm{stat}}(0) = 1$.



Let X_n = position of the ancestor of the individual at the (space-time) origin n generations ago. Given η^{stat} and $X_n = x$, X_{n+1} is uniform on

$$\{y \in \mathbb{Z}^d : y - x \in U,\ \eta_{-n-1}^{\mathrm{stat}}(y) = 1\} \quad (\neq \emptyset).$$

To avoid lots of −-signs later, put $\xi_n(x) := \eta_{-n}^{\mathrm{stat}}(x)$, $x \in \mathbb{Z}^d, n \in \mathbb{Z}$.

Note: $\xi_n(x) = 1$ iff "$(x, n) \to \mathbb{Z}^d \times \{+\infty\}$"

Directed random walk on the supercritical directed cluster

$\omega(x, n)$, $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$, i.i.d. Bernoulli(p), $p > p_c$
$\xi_n(x) = 1$ iff $(x, n) \to (y, k)$ for infinitely many (y, k) $\quad$ ("$(x, n) \to +\infty$")
Write $\mathcal{C} := \{(y, m) : \xi_m(y) = 1\}$, $U(x, n) := (x + U) \times \{n + 1\}$

Directed random walk on the supercritical directed cluster

$\omega(x, n)$, $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$, i.i.d. Bernoulli(p), $p > p_c$
$\xi_n(x) = 1$ iff $(x, n) \to (y, k)$ for infinitely many (y, k) $\quad$ ("$(x, n) \to +\infty$")
Write $\mathcal{C} := \{(y, m) : \xi_m(y) = 1\}$, $U(x, n) := (x + U) \times \{n + 1\}$

Let $X_0 = 0\,(\in \mathbb{Z}^d)$,

$$\mathbb{P}(X_{n+1} = y \mid \xi, X_n = x, X_{n-1} = x_{n-1}, \dots X_1 = x_1) = \frac{\mathbf{1}(y \in U(x, n) \cap \mathcal{C})}{|U(x, n) \cap \mathcal{C}|}$$

(with some arbitrary setting if $U(x, n) \cap \mathcal{C} = \emptyset$, we will later consider ξ under $\mathbb{P}(\cdot \mid (0, 0) \in \mathcal{C})$)

Directed random walk on the supercritical directed cluster

$\omega(x,n)$, $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$, i.i.d. Bernoulli(p), $p > p_c$
$\xi_n(x) = 1$ iff $(x,n) \to (y,k)$ for infinitely many (y,k) ("$(x,n) \to +\infty$")
Write $\mathcal{C} := \{(y,m) : \xi_m(y) = 1\}$, $U(x,n) := (x+U) \times \{n+1\}$

Let $X_0 = 0\,(\in \mathbb{Z}^d)$,

$$\mathbb{P}(X_{n+1} = y \mid \xi,\, X_n = x, X_{n-1} = x_{n-1}, \dots X_1 = x_1) = \frac{\mathbf{1}(y \in U(x,n) \cap \mathcal{C})}{|U(x,n) \cap \mathcal{C}|}$$

(with some arbitrary setting if $U(x,n) \cap \mathcal{C} = \emptyset$, we will later consider ξ under $\mathbb{P}(\cdot \mid (0,0) \in \mathcal{C})$)

Aim: Understand the long-time behaviour of (X_n). Is it similar to "ordinary" random walk?

Remark.
(X_n) is a random walk in space-time random environment (given by ξ).

Random walks in random environments and recently also random walk in space-time random environments have received considerable attention (see e.g. Firas Rassoul-Agha's homepage `http://www.math.utah.edu/~firas/Research/`)

As far as we know, none of the general techniques developed so far in this context is applicable:

- (X_n) is not uniformly elliptic.
- ξ is complicated: not i.i.d., nor is $(\xi_n(x))_{n=0,1,\dots}$ for fixed x a Markov chain.
- The abstract conditions from Dolgopyat, Keller and Liverani (2008) appear very hard to verify.
- The cone-mixing condition from Avena, den Hollander, and Redig (2010) is violated.

Ancestor ordering

For $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$ let $\widetilde{\omega}(x,n) = \big(\widetilde{\omega}(x,n)[1], \widetilde{\omega}(x,n)[2], \dots, \widetilde{\omega}(x,n)[|U|]\big)$ an independent uniform permutation of $U(x,n) = (x+U) \times \{n+1\}$.

Ancestor ordering

For $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$ let $\widetilde{\omega}(x,n) = \big(\widetilde{\omega}(x,n)[1], \widetilde{\omega}(x,n)[2], \dots, \widetilde{\omega}(x,n)[|U|]\big)$ an independent uniform permutation of $U(x,n) = (x+U) \times \{n+1\}$.

$\Gamma^k_{(x,n)}$:= set of all k-step (directed) paths
$$\gamma = \big((x_0,n),(x_1,n+1),\dots,(x_k,n+k)\big)$$

starting at $x_0 = x$ whose steps *begin* at open sites, i.e., $\omega(x_i, n+i) = 1$ for $i = 0, 1, \dots, k-1$.

Ancestor ordering

For $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$ let $\widetilde{\omega}(x,n) = \big(\widetilde{\omega}(x,n)[1], \widetilde{\omega}(x,n)[2], \dots, \widetilde{\omega}(x,n)[|U|]\big)$ an independent uniform permutation of $U(x,n) = (x+U) \times \{n+1\}$.

$\Gamma^k_{(x,n)}$:= set of all k-step (directed) paths

$$\gamma = \big((x_0,n),(x_1,n+1),\dots,(x_k,n+k)\big)$$

starting at $x_0 = x$ whose steps *begin* at open sites, i.e., $\omega(x_i, n+i) = 1$ for $i = 0, 1, \dots, k-1$.

Order $\Gamma^k_{(x,n)} \ni \gamma, \gamma' = \big((x = x'_0, n), (x'_1, n+1), \dots, (x'_k, n+k)\big)$:
$1 \le \ell \,(< k)$ the minimal value s.th. $x_\ell \neq x'_\ell$, then

$$\gamma \prec \gamma' \quad \text{if } x_\ell \text{ has a smaller index than } x'_\ell \text{ in } \widetilde{\omega}(x_{\ell-1}, n+\ell-1).$$

$A^{(1)}_{(x,n);k}$:= (spatial) endpoint of the smallest path in $\Gamma^k_{(x,n)}$ (if $\Gamma^k_{(x,n)} \neq \emptyset$)
(*first* (potential) *ancestor* k generations ago of site (x,n))

Ancestor ordering

For $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$ let $\widetilde{\omega}(x,n) = \big(\widetilde{\omega}(x,n)[1], \widetilde{\omega}(x,n)[2], \dots, \widetilde{\omega}(x,n)[|U|]\big)$ an independent uniform permutation of $U(x,n) = (x+U) \times \{n+1\}$.

$\Gamma^k_{(x,n)}$:= set of all k-step (directed) paths

$$\gamma = \big((x_0,n),(x_1,n+1),\dots,(x_k,n+k)\big)$$

starting at $x_0 = x$ whose steps *begin* at open sites, i.e., $\omega(x_i, n+i) = 1$ for $i = 0, 1, \dots, k-1$.

Order $\Gamma^k_{(x,n)} \ni \gamma, \gamma' = \big((x = x'_0, n), (x'_1, n+1), \dots, (x'_k, n+k)\big)$:
$1 \le \ell \,(< k)$ the minimal value s.th. $x_\ell \neq x'_\ell$, then
$\gamma \prec \gamma'$ if x_ℓ has a smaller index than x'_ℓ in $\widetilde{\omega}(x_{\ell-1}, n+\ell-1)$.

$A^{(1)}_{(x,n);k}$:= (spatial) endpoint of the smallest path in $\Gamma^k_{(x,n)}$ (if $\Gamma^k_{(x,n)} \neq \emptyset$)
(*first* (potential) *ancestor* k generations ago of site (x,n))

Remarks. 1) Construction measurable w.r.t.
$\sigma\big(\omega(y,i), \widetilde{\omega}(y,i) : y \in \mathbb{Z}^d, n \le i < n+k\big)$
2) Discrete time analogue of Neuhauser (1992)

Ancestor ordering and regeneration

$$\kappa(x,n) := \widetilde{\omega}(x,n)\big[\min\{i : \xi_{n+1}(\widetilde{\omega}(x,n)[i]) = 1\} \wedge |U|\big] \quad \text{(with } \min\emptyset := +\infty\text{)}$$

$\kappa(x,n)$ is uniformly distributed on $U(x,n) \cap \mathcal{C}$ if the latter is not empty and uniformly distributed on $U(x,n)$ otherwise.

Ancestor ordering and regeneration

$\kappa(x,n) := \widetilde{\omega}(x,n)\big[\min\{i : \xi_{n+1}(\widetilde{\omega}(x,n)[i]) = 1\} \wedge |U|\big]$ (with $\min \emptyset := +\infty$)

$\kappa(x,n)$ is uniformly distributed on $U(x,n) \cap \mathcal{C}$ if the latter is not empty and uniformly distributed on $U(x,n)$ otherwise.

On $\mathcal{A}_0 := \{(0,0) \in \mathcal{C}\}$

$$X_0 = 0, \quad X_{n+1} := \kappa(X_n, n),\ n = 1, 2, \dots$$

is (a version of) the directed random walk on $\mathcal{C}$, and $X_k = A^{(1)}_{(0,0);k}$ if $\xi_k(A^{(1)}_{(0,0);k}) = 1$.

Ancestor ordering and regeneration

$\kappa(x,n) := \widetilde{\omega}(x,n)\big[\min\{i : \xi_{n+1}(\widetilde{\omega}(x,n)[i]) = 1\} \wedge |U|\big]$ (with $\min\emptyset := +\infty$)

$\kappa(x,n)$ is uniformly distributed on $U(x,n) \cap \mathcal{C}$ if the latter is not empty and uniformly distributed on $U(x,n)$ otherwise.

On $\mathcal{A}_0 := \{(0,0) \in \mathcal{C}\}$

$$X_0 = 0, \quad X_{n+1} := \kappa(X_n, n),\ n = 1, 2, \dots$$

is (a version of) the directed random walk on $\mathcal{C}$, and $X_k = A^{(1)}_{(0,0);k}$ if $\xi_k(A^{(1)}_{(0,0);k}) = 1$.

Regeneration times:

$T_0 = 0,\ Y_0 = 0,$
$T_1 = \min\{n > 0 : \xi_n(A^{(1)}_{(0,0;n)}) = 1\},\ Y_1 = A^{(1)}_{(0,0);n},$
then $T_2 = \min\{n > 0 : \xi_{T_1+n}(A^{(1)}_{(Y_1,T_1);n}) = 1\}$, etc.

Proposition

$\big((Y_i - Y_{i-1}, T_i - T_{i-1})\big)_{i \geq 1}$ is i.i.d. under $\mathbb{P}(\cdot \mid \mathcal{A}_0)$, Y_1 is symmetrically distributed. There exist $C, c \in (0, \infty)$, such that

$$\mathbb{P}(||Y_1|| > n \mid \mathcal{A}_0), \mathbb{P}(\tau_1 > n \mid \mathcal{A}_0) \leq Ce^{-cn} \quad \text{for } n \in \mathbb{N}.$$

Proposition

$\big((Y_i - Y_{i-1}, T_i - T_{i-1})\big)_{i\geq 1}$ is i.i.d. under $\mathbb{P}(\cdot \mid \mathcal{A}_0)$, Y_1 is symmetrically distributed. There exist $C, c \in (0, \infty)$, such that

$$\mathbb{P}(||Y_1|| > n \mid \mathcal{A}_0), \mathbb{P}(\tau_1 > n \mid \mathcal{A}_0) \leq Ce^{-cn} \quad \text{for } n \in \mathbb{N}.$$

Remark

Regeneration structure and proof analogous to Kuczek (1989) and adaptation by Neuhauser (1992):
For tails of $T_1 - T_0$ use "restart" argument (to remove conditioning on $\mathcal{A}_0$) and the fact that finite clusters are small,
i.i.d. property follows from the fact that the ancestor ordering construction uses disjoint time-slices.

LLN and annealed CLT for directed walk on the cluster

Corollary

$$\mathbb{P}\Big(\frac{1}{n}X_n \to 0 \,\Big|\, \omega\Big) = 1 \quad \text{for } \mathbb{P}(\,\cdot \mid \mathcal{A}_0)\text{-a.a. } \omega, \quad \text{and}$$

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{1}{\sqrt{n}}X_n \le x \,\Big|\, \mathcal{A}_0\Big) = \Phi(x) \quad \text{for } x \in \mathbb{R}^d,$$

with Φ the distribution function of a non-trivial d-dimensional normal law.

Two walks on the same cluster

(X_n), (X'_n) two independent directed walks on the same supercritical directed cluster (i.e. using the same ω's, but independent $\widetilde{\omega}$'s resp. $\widetilde{\omega}'$.)

Hopeful theorem in progress ...

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{1}{\sqrt{n}}X_n \le x, \frac{1}{\sqrt{n}}X_n \le x' \,\Big|\, \mathcal{A}_0\Big) = \Phi(x)\Phi(x') \quad \text{for } x, x' \in \mathbb{R}^d,$$

which implies $\mathbb{P}\Big(\frac{1}{\sqrt{n}}X_n \le x \,\Big|\omega\Big) \to \Phi(x) \quad \text{in } L^2\big(\mathbb{P}(\cdot \mid \mathcal{A}_0)\big)$.

Two walks on the same cluster

(X_n), (X'_n) two independent directed walks on the same supercritical directed cluster (i.e. using the same ω's, but independent $\widetilde{\omega}$'s resp. $\widetilde{\omega}'$.)

Hopeful theorem in progress ...

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{1}{\sqrt{n}}X_n \le x, \frac{1}{\sqrt{n}}X_n \le x' \,\Big|\, \mathcal{A}_0\Big) = \Phi(x)\Phi(x') \quad \text{for } x, x' \in \mathbb{R}^d,$$

which implies $\mathbb{P}\Big(\frac{1}{\sqrt{n}}X_n \le x \,\Big|\omega\Big) \to \Phi(x)$ in $L^2(\mathbb{P}(\cdot \mid \mathcal{A}_0))$.

Remarks

1) Quantitative strengthening may allow an a.s. CLT for (X_n)
2) Variation where (X_n) and (X'_n) coalesce upon meeting is of (great) interest in mathematical population genetics
3) (Some) analogous arguments for the continuous-time case by Neuhauser (1992) and Valesin (2010).

A spatial logistic model

Particles "live" in $\mathbb{Z}^d$ in discrete generations,
$\eta_n(x) = \#$ particles at $x \in \mathbb{Z}^d$ in generation n.

Given η_n,

each particle at x has Poisson$\big(m - \sum_z \lambda_{z-x}\eta_n(z)\big)\big)_+$ offspring,
$m > 1$, $\lambda_z \geq 0$, $\lambda_0 > 0$, finite range.

Children take an independent random walk step to y with probability p_{y-x},
$p_{xy} = p_{y-x}$ symmetric, aperiodic finite range random walk kernel on $\mathbb{Z}^d$.

A spatial logistic model

Particles "live" in $\mathbb{Z}^d$ in discrete generations,
$\eta_n(x) = \#$ particles at $x \in \mathbb{Z}^d$ in generation n.

Given η_n,

each particle at x has Poisson$\big(m - \sum_z \lambda_{z-x}\eta_n(z))\big)_+$ offspring,
$m > 1$, $\lambda_z \geq 0$, $\lambda_0 > 0$, finite range.

Children take an independent random walk step to y with probability p_{y-x},
$p_{xy} = p_{y-x}$ symmetric, aperiodic finite range random walk kernel on $\mathbb{Z}^d$.

Given η_n,

$$\eta_{n+1}(y) \sim \mathrm{Poi}\Big(\sum_x p_{y-x}\eta_n(x)\Big(m - \textstyle\sum_z \lambda_{z-x}\eta_n(z)\Big)_+\Big), \quad \text{independent}$$

Survival and complete convergence

Theorem (B. & Depperschmidt, 2007)

Assume $m \in (1,3)$, $0 < \lambda_0 \ll 1$, $\lambda_z \ll \lambda_0$ for $z \neq 0$.

(η_n) survives for all time globally and locally with positive probability for any non-trivial initial condition η_0.

Given survival, η_n converges in distribution to its unique non-trivial equilibrium.

Starting from any two initial conditions η_0, η'_0, copies (η_n), (η'_n) can be coupled such that if both survive, $\eta_n(x) = \eta'_n(x)$ in a space-time cone.

Survival and complete convergence

Theorem (B. & Depperschmidt, 2007)

Assume $m \in (1,3)$, $0 < \lambda_0 \ll 1$, $\lambda_z \ll \lambda_0$ for $z \neq 0$.

(η_n) survives for all time globally and locally with positive probability for any non-trivial initial condition η_0.

Given survival, η_n converges in distribution to its unique non-trivial equilibrium.

Starting from any two initial conditions η_0, η_0', copies (η_n), (η_n') can be coupled such that if both survive, $\eta_n(x) = \eta_n'(x)$ in a space-time cone.

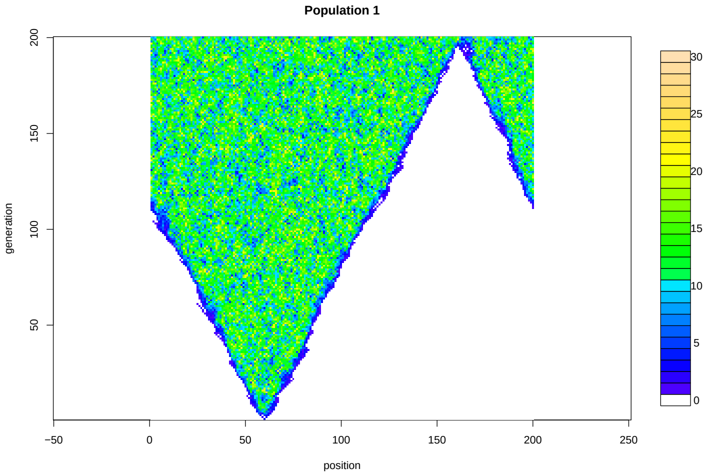
Proof uses that corresponding deterministic system

$$\zeta_{n+1}(y) = \sum_x p_{y-x}\zeta_n(x)\Big(m - \sum\nolimits_z \lambda_{z-x}\zeta_n(z)\Big)_+$$

has unique non-triv. fixed point

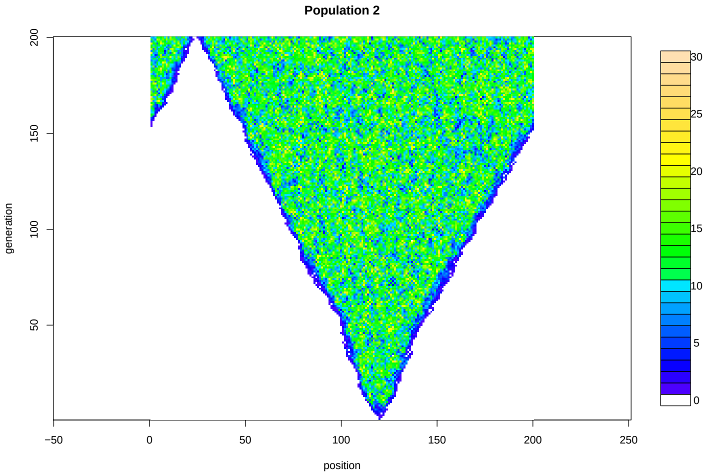
plus coarse-graining, lots of comparisons with directed percolation.

Coupling



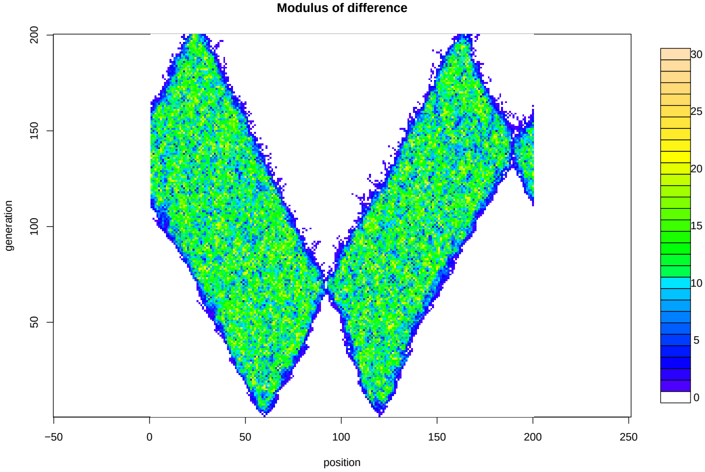
$m = 1.5,\ p = (1/3, 1/3, 1/3),\ \lambda = (0.01, 0.02, 0.01)$

Coupling



$m = 1.5,\ p = (1/3, 1/3, 1/3),\ \lambda = (0.01, 0.02, 0.01)$

Coupling



$m = 1.5,\ p = (1/3, 1/3, 1/3),\ \lambda = (0.01, 0.02, 0.01)$

Ancestral lines

Given stationary $(\eta_n^{\text{stat}}(x), n \in \mathbb{Z}, x \in \mathbb{Z}^d)$, cond. on $\eta_0^{\text{stat}}(0) > 0$, sample an individual from space-time origin $(0, 0)$ (uniformly)

Let (X_n) position of her ancestor n generations ago:

Given η^{stat} and $X_n = x$, $X_{n+1} = y$ w. prob.

$$\frac{p_{x-y}\eta_{-n-1}^{\text{stat}}(y)\Big(m - \sum_z \lambda_{z-y}\eta_{-n-1}^{\text{stat}}(z)\Big)_+}{\sum_{y'} p_{x-y'}\eta_{-n-1}^{\text{stat}}(y')\Big(m - \sum_z \lambda_{z-y'}\eta_{-n-1}^{\text{stat}}(z)\Big)_+}$$

Ancestral lines

Given stationary $(\eta_n^{\text{stat}}(x), n \in \mathbb{Z}, x \in \mathbb{Z}^d)$, cond. on $\eta_0^{\text{stat}}(0) > 0$, sample an individual from space-time origin $(0,0)$ (uniformly)

Let (X_n) position of her ancestor n generations ago:

Given η^{stat} and $X_n = x$, $X_{n+1} = y$ w. prob.

$$\frac{p_{x-y}\eta^{\text{stat}}_{-n-1}(y)\Big(m - \sum_z \lambda_{z-y}\eta^{\text{stat}}_{-n-1}(z)\Big)_+}{\sum_{y'} p_{x-y'}\eta^{\text{stat}}_{-n-1}(y')\Big(m - \sum_z \lambda_{z-y'}\eta^{\text{stat}}_{-n-1}(z)\Big)_+}$$

Hopeful theorem in progress ...

If $m \in (1,3)$, $0 < \lambda_0 \ll 1$, $\lambda_z \ll \lambda_0$ for $z \neq 0$, there is a regeneration construction for (X_n).

Thank you for your attention!