Convergence of Rescaled Competing Species Processes to a Class of SPDEs

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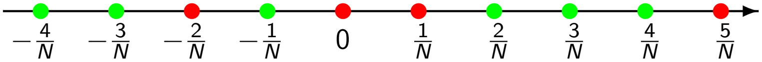
In the N^{th} model:

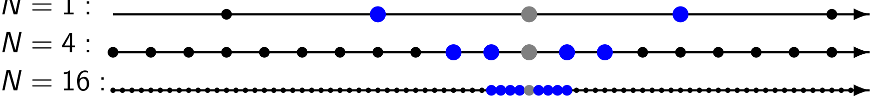
- space: $\mathbb{Z}/N$,
- state-space of each site $x \in \mathbb{Z}/N$: $\{0, 1\}$ respectively { , }.
  Think of: individual with political opinion 0 or 1
  or: two populations 0 and 1.
- state of the system at time t: $\xi_t^N : \mathbb{Z}/N \to \{0, 1\}$, i.e.
  $\xi_t^N(x)$ gives state of x at time t:



i.e. $\xi_t^N(-2/N) =$, $\xi_t^N(3/N) =$, etc.

The dynamics of the process

- neighbours of x: $y \sim x$ iff $0 < |x - y| \leq N^{-1/2}$



Each x has $2c(N)N^{1/2}$, $c(N) \stackrel{N\to\infty}{\to} 1$ neighbours.

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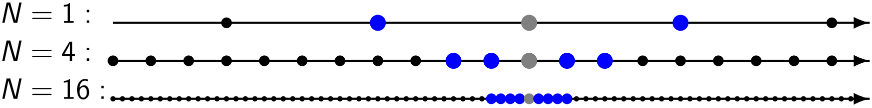

Each x has $2c(N)N^{1/2}$, $c(N) \overset{N\to\infty}{\to} 1$ neighbours.

Long-range interaction takes into account the densities of the neighbours of $x \in \mathbb{Z}/N$ at long-range, i.e.

$$f_i^{(N)}(x,\xi) \equiv \frac{1}{|\{y : y \sim x\}|} \sum_{y:y\sim x} 1(\xi^N(y) = i), \quad i = 0, 1.$$

Note in particular:

- $0 \leq f_i^{(N)} \leq 1$ and
- $f_0^{(N)} + f_1^{(N)} = 1$.

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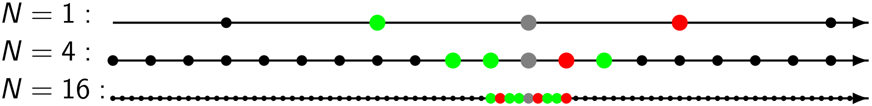

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- $0 \le f_i^{(N)} \le 1$ and
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The evolution of the process in time is given via infinitesimal rates. $c(x, \xi)$ denotes the rate at which the coordinate $\xi(x)$ flips from 0 to 1 or from 1 to 0 when the system is in state ξ. Then the process ξ_t will satisfy

$$\mathbb{P}(\xi_t(x) \neq \xi_0(x)) = c(x, \xi_0)t + o(t) \text{ for } t \downarrow 0^+.$$

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Flip rates of the *unscaled voter process*:

$$0 \to 1 \text{ at rate } c(x, \xi) = f_1(x, \xi),$$
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Flip rates of the *unscaled biased voter process*:

$$0 \to 1 \text{ at rate } c(x, \xi) = (1 + \tau) f_1(x, \xi),$$
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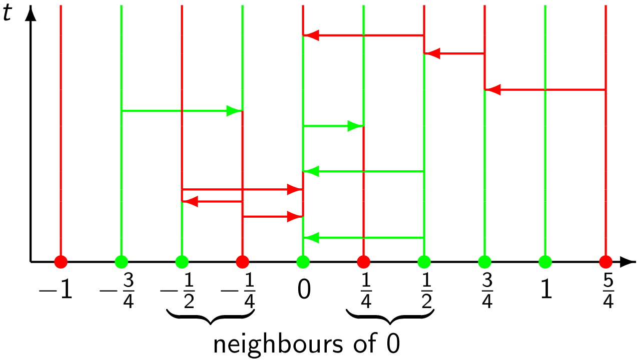
Graphical representation of the long-range voter process

Example: $N = 4$



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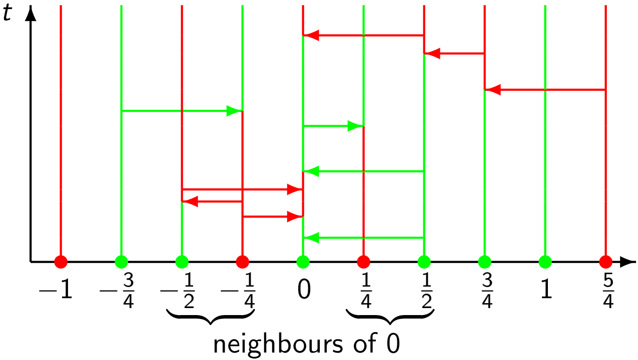

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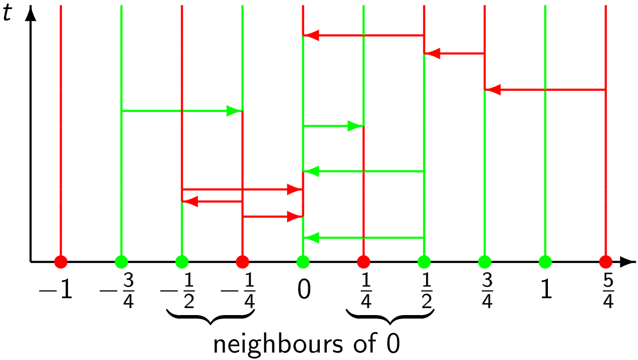

- rate of an arrow to 0 from $\frac{1}{2}$: $\frac{1}{\text{"no. of neighbours"}} = \frac{1}{4}$
- rate of an arrow to site 0: $\frac{1}{4} \times \text{"no. of neighbours"} = 1$

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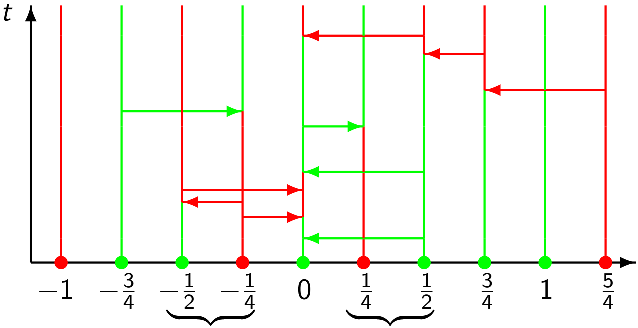

- rate of an arrow to 0 from $\frac{1}{2}$: $\frac{1}{\text{"no. of neighbours"}} = \frac{1}{4}$
- rate of an arrow to site 0: $\frac{1}{4} \times \text{"no. of neighbours"} = 1$
- rate of an arrow to 0, at time $t = 0$, that changes colour at 0: $\frac{1}{\text{"no. of neighbours"}} \times \text{"no. of red neighbours"} = \frac{1}{2}$

Recall:

- Flip rates of the *unscaled biased voter process*:

$$\begin{aligned} 0 \to 1 \text{ at rate } c(x,\xi) &= (1+\tau) f_1(x,\xi), \\ 1 \to 0 \text{ at rate } c(x,\xi) &= f_0(x,\xi). \end{aligned}$$

Recall:

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$$0 \to 1 \text{ at rate } c(x,\xi) = (1+\tau) f_1(x,\xi),$$
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- Rescaling for the *biased voter process*::

$$\begin{aligned} 0 \to 1 \text{ at rate } c(x,\xi) &= N\left(1+\frac{\tau}{N}\right) f_1^{(N)}(x,\xi) \\ &= N f_1^{(N)}(x,\xi) + f_1^{(N)}(x,\xi)\tau, \\ 1 \to 0 \text{ at rate } c(x,\xi) &= N f_0^{(N)}(x,\xi). \end{aligned}$$

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- Adding more general perturbations:

$$0 \to 1 \text{ at rate } N f_1^{(N)} + f_1^{(N)} G_0^{(N)}\left(f_1^{(N)}\right),$$
$$1 \to 0 \text{ at rate } N f_0^{(N)} + f_0^{(N)} G_1^{(N)}\left(f_0^{(N)}\right),$$

where $G_i^{(N)}, i = 0, 1$ are power series on $[0,1]$, i.e.

$$G_i^{(N)}(x) = \sum_{m=0}^{\infty} \alpha_i^{(m+1,N)} x^m, \quad i = 0, 1, x \in [0, 1]$$

with $\alpha_i^{(m+1,N)}$ satisfying certain summability and convergence conditions, uniformly in $N \geq N_0$. As a result define

$$G_i(x) \equiv \lim_{N\to\infty} G_i^{(N)}(x) = \sum_{m=0}^{\infty} \lim_{N\to\infty} \alpha_i^{(m+1,N)} x^m = \sum_{m=0}^{\infty} \alpha_i^{(m+1)} x^m$$

for $x \in [0, 1]$.

The object of interest

Approximate density $A(\xi_t^N)$ for the configurations ξ_t^N:

$$A(\xi_t^N)(x) = \frac{1}{|\{y : y \sim x\}|} \sum_{y:y\sim x} \xi_t^N(y), \qquad x \in \mathbb{Z}/N.$$

Note: $A(\xi_t^N)(x) = f_1^{(N)}(x, \xi_t^N)$.

By linearly interpolating between sites we obtain approximate densities $A(\xi_t^N)(x) \in [0,1]$ for all $x \in \mathbb{R}$.

Notation

Set $\mathcal{C}_1 \equiv \{f : \mathbb{R} \to [0,1]$ *continuous*$\}$ *and let* $\mathcal{C}_1$ *be equipped with the topology of uniform convergence on compact sets.*

We obtain that $t \mapsto A(\xi_t^N)$ is càdlàg $\mathcal{C}_1$-valued.

Theorem

Suppose that $A(\xi_0^N) \to u_0$ *in* $\mathcal{C}_1$ *and that* $G_i^{(N)}$, $i = 0, 1$ *satisfy appropriate Hypotheses. Then*

- $\big(A(\xi_t^N) : t \geq 0\big)$ *are C-tight as càdlàg* $\mathcal{C}_1$*-valued processes.*
- *The limit points of* $A(\xi_t^N)$ *are continuous* $\mathcal{C}_1$*-valued processes* u_t *which solve*

$$\frac{\partial u}{\partial t} = \frac{\Delta u}{6} + (1-u)u\left\{G_0(u) - G_1(1-u)\right\} + \sqrt{2u(1-u)}\dot{W}$$

  with initial condition u_0.
- *If we assume additionally* $\int u_0(x)dx < \infty$, *then* u_t *is the unique in law* $[0, 1]$*-valued solution to the above SPDE.*

Example 1: The Lotka-Volterra model

$$0 \to 1 \text{ at rate } c(x,\xi) = f_1(x,\xi)\left(f_0(x,\xi) + a_{01} f_1(x,\xi)\right)$$

$$1 \to 0 \text{ at rate } c(x,\xi) = f_0(x,\xi)\left(f_1(x,\xi) + a_{10} f_0(x,\xi)\right)$$

- The first factor of the rate represents the strength of the instantaneous replacement by a particle of opposite type.
- The second factor of the rate governs the density-dependent mortality of a particle.
  - f_0 describes the effect of intraspecific competition,
  - $a_{01} f_1$ the effect of interspecific competition.

Rewrite, using $f_0 + f_1 = 1$,

$$\begin{aligned} 0 \to 1 \text{ at rate } c(x,\xi) &= f_1(x,\xi)\left(f_0(x,\xi) + a_{01} f_1(x,\xi)\right) \\ &= f_1(x,\xi)\left(1 + (a_{01}-1) f_1(x,\xi)\right) \end{aligned}$$

$$\begin{aligned} 1 \to 0 \text{ at rate } c(x,\xi) &= f_0(x,\xi)\left(f_1(x,\xi) + a_{10} f_0(x,\xi)\right) \\ &= f_0(x,\xi)\left(1 + (a_{10}-1) f_0(x,\xi)\right). \end{aligned}$$

If we choose a_{01}, a_{10} close to 1, the Lotka-Volterra model can be seen as a small perturbation of the voter model.

Consider a sequence of rescaled Lotka-Volterra models with rates of change

$$0 \to 1 \text{ at rate } Nf_1^{(N)} \left(1 + \left(a_{01}^{(N)} - 1\right) f_1^{(N)}\right),$$
$$1 \to 0 \text{ at rate } Nf_0^{(N)} \left(1 + \left(a_{10}^{(N)} - 1\right) f_0^{(N)}\right).$$

For $i = 0, 1$ *choose*

$$a_{i(1-i)}^{(N)} - 1 \equiv \frac{\theta_i^{(N)}}{N} \text{ with } \theta_i^{(N)} \overset{N \to \infty}{\to} \theta_i$$

and rewrite

$$0 \to 1 \text{ at rate } Nf_1^{(N)} + \theta_0^{(N)} \left(f_1^{(N)}\right)^2 = Nf_1^{(N)} + f_1^{(N)} \theta_0^{(N)} f_1^{(N)},$$
$$1 \to 0 \text{ at rate } Nf_0^{(N)} + \theta_1^{(N)} \left(f_0^{(N)}\right)^2 = Nf_0^{(N)} + f_0^{(N)} \theta_1^{(N)} f_0^{(N)}.$$

The Theorem yields that the sequence of approximate densities $A(\xi_t^N)$ is tight and every solution solves

$$\frac{\partial u}{\partial t} = \frac{\Delta u}{6} + (1-u)u\left\{\theta_0 u - \theta_1(1-u)\right\} + \sqrt{2u(1-u)}\dot{W}$$

with initial condition u_0. Uniqueness in law holds for initial conditions of finite mass.

Literature Review

- This paper is an extension of results of Mueller and Tribe [3] ($d = 1$, voter processes with nonnegative bias).

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- This paper is an extension of results of Mueller and Tribe [3] ($d = 1$, voter processes with nonnegative bias).
- In Cox and Perkins [1] it was shown that rescaled Lotka-Volterra models with long-range interaction converge weakly to super-Brownian motion with linear drift. They consider
  - low density regime
  - **weak** limits for measure-valued processes

$$X_t^N = \frac{1}{N} \sum_{x \in \mathbb{Z}/(M_N\sqrt{N})} \xi_t^N(x)\delta_x$$

  with $M_N/\sqrt{N} \to \infty$ (for $d = 1$)
  - We consider $M_N = \sqrt{N}$ (we also get X_t^N converges to $u_t dt$ in the **vague** topology).

- Additionally, [1] consider fixed kernel models in dimensions $d \geq 2$ respectively $d \geq 3$. In Cox and Perkins [2], the results of [1] for $d \geq 3$ are used to relate the limiting super-Brownian motions to questions of coexistence and survival of a rare type in the original Lotka-Volterra model.

Example 2

Consider rescaled Lotka-Volterra models with long-range dispersal and short-range competition, i.e. where

$$0 \to 1 \textit{ at rate } Nf_1^{(N)}\left(g_0^{(N)} + a_{01}^{(N)} g_1^{(N)}\right),$$
$$1 \to 0 \textit{ at rate } Nf_0^{(N)}\left(g_1^{(N)} + a_{10}^{(N)} g_0^{(N)}\right).$$

Here $f_i^{(N)}, i = 0, 1$ *is the density corresponding to a long-range kernel and* $g_i^{(N)}, i = 0, 1$ *is the density corresponding to a fixed kernel.*

Example 3

Spatial versions of the Lotka-Volterra model

Introduced in Neuhauser and Pacala [4]. Consider

$$0 \to 1 \text{ at rate } N\left[\frac{\lambda^{(N)} f_1^{(N)}}{\lambda^{(N)} f_1^{(N)} + f_0^{(N)}}\left(f_0^{(N)} + a_{01}^{(N)} f_1^{(N)}\right)\right],$$

$$1 \to 0 \text{ at rate } N\left[\frac{f_0^{(N)}}{\lambda^{(N)} f_1^{(N)} + f_0^{(N)}}\left(f_1^{(N)} + a_{10}^{(N)} f_0^{(N)}\right)\right].$$

Choose competition parameters and fecundity parameter λ near one:

$$\lambda^{(N)} \equiv 1 + \frac{\lambda'}{N},\ a_{01}^{(N)} \equiv 1 + \frac{a_{01}}{N},\ a_{10}^{(N)} \equiv 1 + \frac{a_{10}}{N}.$$

The limit points of $A(\xi_t^N)$, u_t solve

$$\frac{\Delta u}{6} + (1-u)u\left\{\lambda' - a_{10} + u\left(a_{01} + a_{10}\right)\right\} + \sqrt{2u(1-u)}\dot{W}.$$

Example 4

We obtain a class of SPDEs,

$$\frac{\partial u}{\partial t} = \frac{\Delta u}{6} + (1-u)u\left\{G_0(u) - G_1(1-u)\right\} + \sqrt{2u(1-u)}\dot{W}$$

with $u_0 \in \mathcal{C}_1$*, that can be characterized as the limit of perturbations of the long-range voter model.*

Proof of the Theorem

Proof Part 1: "How to get positive perturbations only."

Recall:

$$0 \to 1 \text{ at rate } Nf_1 + f_1 \sum_{m=0}^{\infty} \alpha_0^{(m+1,N)} f_1^m,$$

$$1 \to 0 \text{ at rate } Nf_0 + f_0 \sum_{m=0}^{\infty} \alpha_1^{(m+1,N)} f_0^m.$$

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Rewrite the rates in a form, where all resulting coefficients are non-negative by using

$$-x^m = (1-x) \sum_{l=1}^{m-1} x^l - x \quad \text{and} \quad 1 - f_1 = f_0.$$

Lemma

We obtain

$$0 \to 1 \text{ at rate } (N-\theta) f_1 + f_1 \sum_{m \geq 2, j=0,1} q_j^{(0,m)} f_j f_1^{m-2}, \tag{1}$$

$$1 \to 0 \text{ at rate } (N-\theta) f_0 + f_0 \sum_{m \geq 2, j=0,1} q_j^{(1,m)} f_j f_0^{m-2},$$

with corresponding $\theta = \theta^{(N)}, q_j^{(k,m)} = q_j^{(k,m,N)} \in \mathbb{R}^+, j, k = 0, 1, m \geq 2.$

Proof Part 2: Tightness and SPDE-limit

Step 1: Graphical construction

Suppose

$$0 \to 1 \text{ at rate } \cdots + q_j^{(0,m)} f_j f_1^{m-1} + \cdots$$

with $j \in \{0,1\}$, $q_j^{(0,m)} > 0$.

Recall: $f_i^{(N)}(x,\xi) \equiv \frac{1}{2c(N)\sqrt{N}} \sum_{y:y\sim x} 1(\xi^N(y) = i)$, $i = 0, 1$.

The graphical construction uses independent families of i.i.d. Poisson processes: E.g.,

$$\left(Q_t^{m,j,0}(x; y_1, \ldots, y_m) : x, y_1, \ldots, y_m \in N^{-1}\mathbb{Z} \right)$$

$$\text{i.i.d. Poisson processes of rate } \frac{q_j^{(0,m)}}{2c(N)\sqrt{N}(2c(N)\sqrt{N})^{m-1}}.$$

At a jump of $Q_t^{m,j,0}(x; y_1, \ldots, y_m)$ the voter at x adopts the opinion 1 provided that $y_1, \ldots, y_m$ are neighbours of x, y_1 has opinion j and all of $y_2 \ldots, y_m$ have opinion 1.

$\Rightarrow$ stochastic integral equation for ξ_t^N:

$$
\begin{aligned}
\xi_t^N(x) =& \xi_0^N(x) \\
&+ \sum_{y \sim x} \int_0^t \left\{ \delta_0\left(\xi_{s-}^N(x)\right) \delta_1\left(\xi_{s-}^N(y)\right) - \delta_1\left(\xi_{s-}^N(x)\right) \delta_0\left(\xi_{s-}^N(y)\right) \right\} \\
&\quad \times dP_s(x; y) \\
&+ \sum_{k=0,1} (1-2k) \sum_{m \geq 2, i, j = 0,1} \sum_{y_1, \ldots, y_m \sim x} \int_0^t \delta_k\left(\xi_{s-}^N(x)\right) \\
&\quad \times \delta_j\left(\xi_{s-}^N(y_1)\right) \prod_{l=2}^{m} \delta_{1-k}\left(\xi_{s-}^N(y_l)\right) dQ_s^{m,j,k}(x; y_1, \ldots, y_m)
\end{aligned}
$$

for all $x \in N^{-1}\mathbb{Z}$.

Step 2: An approximate martingale problem

- *Use:* If $N \sim \text{Pois}(\lambda)$, then $N_t - \lambda t$ is a martingale with quadratic variation $\langle N \rangle_t = \lambda t$.
- Integrate against test-functions $\phi_t(x)$, i.e. calculate $\frac{1}{N} \sum_{x \in \mathbb{Z}/N} \xi_t(x) \phi_t(x)$,

$\Rightarrow$ an approximate semimartingale decomposition for $\frac{1}{N} \sum_{x \in \mathbb{Z}/N} \xi_t^N(x) \phi_t(x)$.

Step 3: Green's function representation for $A(\xi_t^N)$

Choose "clever" test function $\phi_t(x)$
$\Rightarrow$ approximate Green's function representation for $A(\xi_t^N)$.
Note: Taking $N \to \infty$ we find the form of the SPDE.

Step 4: Tightness estimates

Derive estimates on p^{th}-moment differences, i.e. bound (I omit some details here)

$$\mathbb{E}\left[\left|A(\xi_t^N)(z) - A(\xi_s^N)(y)\right|^p\right] \leq Ce^{\lambda p|z|}\left(|t-s|^{p/24} + |z-y|^{p/24} + N^{-p/24}\right).$$

Then use Kolmogorov's continuity theorem and the Arzelà-Ascoli theorem.

Proof Part 3: Uniqueness in law

Apply a version of Dawson's Girsanov theorem.

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thank you