Dynamic networks in dynamic populations

Tom Britton

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Joint work with Mathias Lindholm and Tatyana Turova

Background

Aim: Mimic a social network (of individuals and their friendships) over time

Specific aims:

- new individuals are born into the community
- individuals should eventually die
- at birth individuals are given a "social index"
- "friendships" between individuals can be created
- old frienships may stop
- the social index of an individual affects how many friends he/she makes

Model properties of Barabasi and Albert, 1999:

- individuals have age (time until birth)
- The number of edges of an individual increases (stochastically) with age
- Individuals having same number of friends at some time, has the same distribution of receiving future friends
- The more friends an individual has, the more new friends will he/she get
- The limiting degree distribution (of the number of friends) will be heavy-tailed (power law)

Our new model:
The Markovian network in a Markovian population

The model consists of a population model *and* a model for friendships within the population

Population model:

- One individual alive at $t = 0$
- individuals give birth at rate λ and die at rate μ ($\lambda > \mu$, a super-critical linear birth-and-death process)
- each individual is given a social index S from distribution F_S (i.i.d.)

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The Markovian network in a Markovian population (cont'd)

Network model:

- Markovian given the population
- No friends at birth. An individual with social index s creates new friends at rate αs (independently of everything alse)
- The friend is selected either

  U-version: uniformly among all living individuals, or

  P-version: with prob prop to the social index
- Each friendship ceases (independently) at rate β

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Model parameters:

$\lambda =$ individual birth rate

$\mu =$ individual death rate

$F_S =$ distribution of social index S

$\alpha =$ "rate" at which an individual gets new friends

$\beta =$ rate at which a friendship stops

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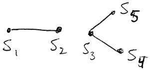
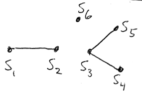
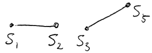
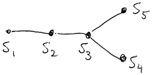
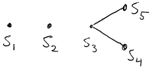
At t (#alive $=Y(t)=y=5$)	At $t+h$	Event	Rate
 →		New individual born	$\lambda\cdot y$
→		individual $\cancel{5}$ 4 dies	μ
→		Friendship between 2 and 3 created	U-version: $\alpha\cdot S_2\cdot\frac{1}{y-1}+\alpha\cdot S_3\cdot\frac{1}{y-1}$ P-version: $\alpha\cdot S_2\cdot\frac{S_3}{S_1+S_3+S_4+S_5}+\alpha\cdot S_3\frac{S_2}{S_1+S_2+S_4+S_5}$
→		Friendship between 1 and 2 stops	β

Results: Population properties

Questions treated: What will the network "look" like after having evolved for a long time: pop size, degree distribution, degree correlation, giant component, ...?

Population properties:

- The population is a super-critical Markovian branching process (a well-studied object)
- $Y(t) :=$ the number of individuals alive at t
- $Y(t) \to \infty$ or $Y(t) \to 0$ as $t \to \infty$
- $Y(t) \sim e^{t(\lambda-\mu)}$ on $B = \{Y(t) \to \infty\}$
- For large t on B, the age A of a random individual is $A \sim \mathrm{Exp}(\lambda)$ (stable-age distribution)

Results: Network properties

Network properties: the degree distribution, U-version:

$X_S(A) :=$ # friends of randomly chosen individual at t (t large)

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Condition on $A = a$ and $S = s$

What is distribution of $X_s(a)$?

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What is distribution of $X_s(a)$?

Answer: $X_s(u),\ 0 \leq u \leq a$ is **birth-and-death process:**

- $X_s(0) = 0$
- *Constant birth rate*: new friendship created at rate $\alpha(s + E(S))$, (second term for receiving edges)
- *Linear death rate*: each friendship lost at rate $\beta + \mu$ (edge or opposite node "dies")

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Network properties: the degree distribution

B-D process theory $\Longrightarrow X_s(a) \sim \mathrm{Po}\left(\frac{\alpha(s+E(S))(1-e^{-(\beta+\mu)a})}{\beta+\mu}\right)$

$\Longrightarrow$ unconditional distribution:

$$X_S(A) \sim \mathrm{MixPo}\left(\frac{\alpha(S+E(S))(1-e^{-(\beta+\mu)A})}{\beta+\mu}\right),$$

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Moments:

$$\begin{aligned} E(X_S(A)) &= \frac{2\alpha}{\beta+\mu+\lambda}E(S) \\ V(X_S(A)) &= c_1(\lambda,\mu,\alpha,\beta)E(S) + c_2(\lambda,\mu,\alpha,\beta)(E(S))^2 \\ &\quad + c_3(\lambda,\mu,\alpha,\beta)V(S) \end{aligned}$$

Tail of degree dist $X_S(A)$ depends on tail of F_S

Network properties: the degree distribution

Network properties: the degree distribution, P-version:

Analysed similarly ...

$$\Longrightarrow X_S(A) \sim \mathrm{MixPo}\left(\frac{2\alpha S(1-e^{-(\beta+\mu)A})}{\beta+\mu}\right),$$

where $S \sim F_S$ and $A \sim \mathrm{Exp}(\lambda)$ are independent

Only difference: $\alpha(S + E(S)) \ \rightarrow \ 2\alpha S$

Moments: Same mean, but larger variance

Network properties: Type distribution of neighbours

What type of friends will (a, s)-individuals have?

It can be shown that $f(a', s'|a, s) = f(a'|a)f(s'|s)$, where

$$f(a'|a) = \frac{(\beta+\mu)\lambda e^{-\lambda a'}\left(e^{(\lambda-\beta-\mu)a\wedge a'} - 1\right)}{(\lambda-\beta-\mu)(1-e^{-(\beta+\mu)a})},$$

for both versions, and

$$f^{(U)}(s'|s) = \frac{(s+s')f_S(s')}{s+E(S)},$$

$$f^{(P)}(s'|s) = \frac{s' f_S(s')}{E(S)}. \quad \text{Independent!}$$

Network properties: Degree correlation ρ

What is correlation ρ between nodes of a randomly selected edge?

Can be obtained in two different ways:

1. Random walk:
   - Take a snapshot of the network after long time
   - Perform a random walk on network using type distribution just derived
   - Look at the type distribution of a node in stationarity
   - Compute degree correlation ρ between this node and a randomly selected neighbour using conditional Poisson distribution $X_s(a)$

Network properties: Degree correlation ρ

2. Size-biased selection of nodes
   - Pick first a random *edge*
   - The degree distribution of its adjacent node will have a size-biased version of the mixed Poisson distribution
   - Identify the type-distribution of this node
   - Compute the degree correlation ρ between this node and one of its neighbours as in first method

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Network properties: Degree correlation ρ and clustering

Conclusions:

Both methods give the same degree correlation ...

U-version: $\rho > 0$ if $V(S)$ small
$\rho < 0$ if $V(S)$ large

P-version: $\rho > 0$ (irrespective of parameter values)

"Explanation":

– Varying **age** induces **positive** degree correlation

– In P-version **social indices** are independent.

– In U-version they induce **negative** degree correlation, more the more skew F_S is.

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Clustering: $c = 0$ ($c =$ limiting fraction of triangles)

Because frienships occur independently of other friendships

Network properties: Phase transition

Is there a giant connected component in the network?

Apply methods for inhomogeneous random graphs (Bollobás, Janson and Riordan, 2007)

A critical parameter R given as implicit solution to certain equation

Giant component exists if $R > 1$

Qualitative conclusion:
R is *increasing* in: $V(S)$, α and λ
R is *decreasing* in β and μ

The degree distribution: Examples

Numerical example: $\mu = 1/80,\ \lambda = 1.1/80,\ \alpha = 1,\ \beta = 0.5$ (Slightly super-critical, 80 years life-lenght, one new friendship per year, friendships last 2 years)

S-dist	$E(X_S(A))$	$V(X_S(A))$
$S \equiv 1$	≈ 3.8	≈ 2.0
$S \sim \mathrm{Exp}(1)$	≈ 3.8	≈ 6.0
$S \sim \Gamma(1/2, 1/2)$	≈ 3.8	≈ 14.3
$S \sim \mathrm{Par}(1/2, 2)$	≈ 3.8	∞

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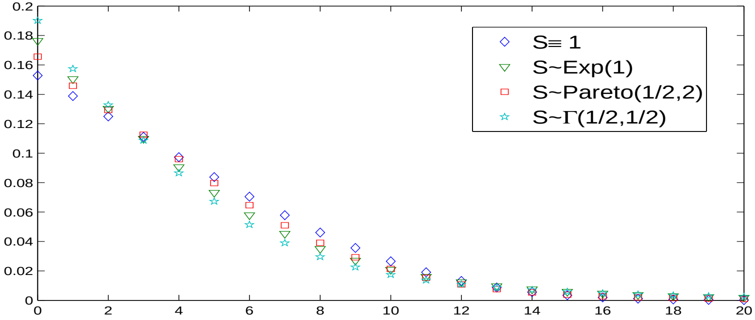
The degree distribution: Examples

The degree distribution: small degrees



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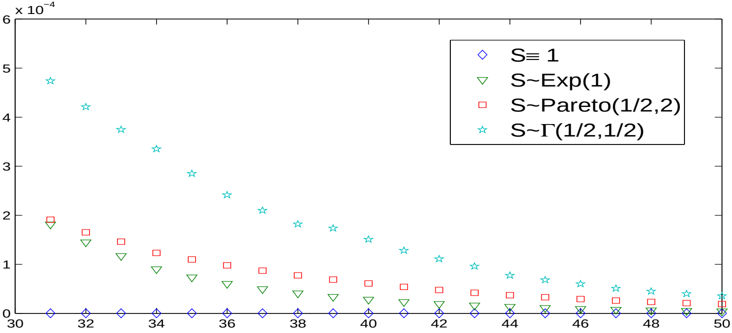
The degree distribution: Examples

The degree distribution: intermediate degrees



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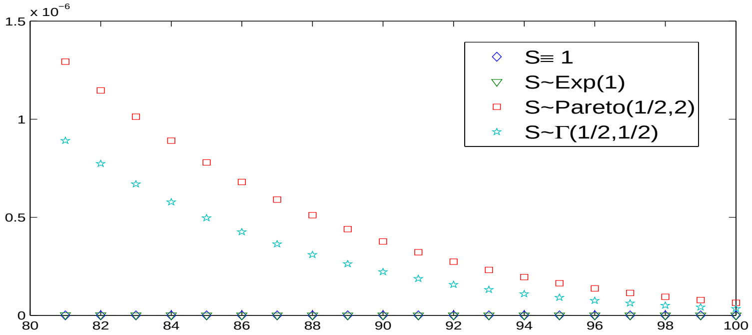
The degree distribution: Examples

The degree distribution: high degrees



The degree distribution: Simulations

Results rely on $t \to \infty$ (and $Y(t) \to \infty$)

How about finite t and finite $Y(t)$?

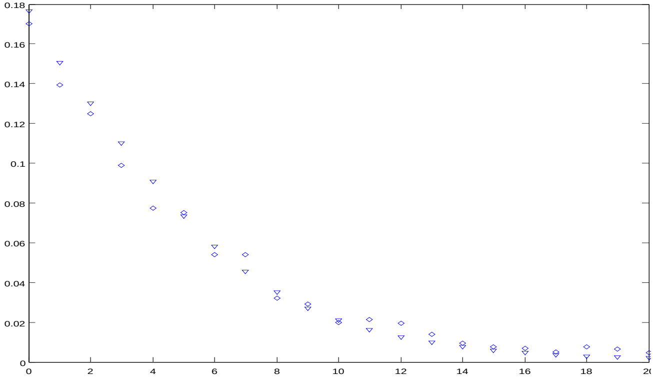
Simulations:
Simulated network until 2000 alive individuals

Comparison of empirical and theoretical degree distributions

The degree distribution: Simulations

Simulations: same parameter values as before, $S \sim \mathrm{Exp}(1)$

Empirical vs theoretical degree distributions



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Future work

Possible model extensions:

Clustering: can be obtained
- by inheriting friendships from mother, or
- by increased birth rate of friendships among friends-of-friends

Social index: distribution of S might depend on mother's S'

Dynamic process "on" the network: What happens if an epidemic spreads in the network?

References

- Britton, T. and Lindholm, M. (2010): Dynamic random networks in dynamic populations. *J. Stat. Phys.*, **139**, 518-535.

- T. Britton, M. Lindholm and T. Turova. (2011): A dynamic network in a dynamic population: asymptotic properties. arXiv:1104.0148