A Superfluid Universe

Lecture 1

General relativity and cosmology

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Lecture 1. General relativity and cosmology

- Mathematics and physics
- Big bang
- Dark energy
- Dark matter
- Robertson-Walker
- Schwarzchild
- Black hole

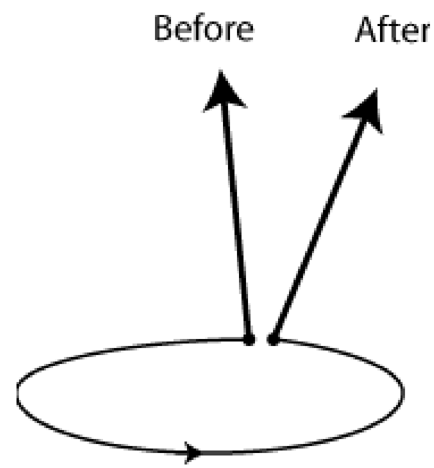
$$x^{\mu} = (t, x, y, z) \quad \text{(c=1)}$$

$$ds^2 = g^{\mu\nu} dx_{\mu} dx_{\nu}$$

Flat space: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$g^{\mu\nu} =$ diagonal (-1, 1, 1, 1)

Space is curved, when vector's direction changes upon "parallel displacement" (covariant derivative=0) in closed loop .



Mathematics

$$D_\mu v^\nu = \frac{\partial}{\partial x^\mu} v^\nu + \Gamma^\nu{}_{\mu\alpha} v^\alpha$$ (Covariant derivative)

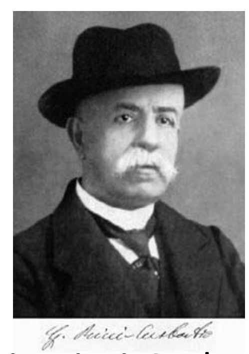
$$\Gamma^\nu{}_{\mu\alpha} = \frac{1}{2} g^{\nu\beta} \left(\frac{\partial}{\partial x^\alpha} g^{\beta\mu} + \frac{\partial}{\partial x^\mu} g^{\beta\alpha} + \frac{\partial}{\partial x^\beta} g^{\mu\alpha} \right)$$ (Connection, or Christoffel symbol)

$$R_{\alpha\beta} = \frac{\partial}{\partial x^\gamma} \Gamma^\gamma{}_{\alpha\beta} - \frac{\partial}{\partial x^\beta} \Gamma^\gamma{}_{\alpha\gamma} + \Gamma^\gamma{}_{\alpha\beta} \Gamma^\delta{}_{\gamma\delta} - \Gamma^\gamma{}_{\alpha\delta} \Gamma^\delta{}_{\beta\gamma}$$ (Curvature tensor)

$$R = R^\alpha{}_\alpha$$ (Scalar curvature)

Bernhard Riemann
(1826-1866)

Elwin Bruno Christoffel
(1829-1900)

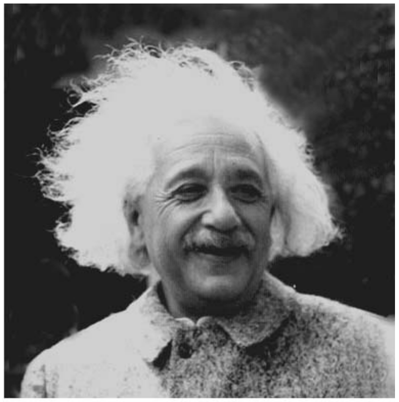


Gregorio Ricci-Curbastro
(1853-1925)

Physics

Matter is source of curvature (gravitational field)

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta} \quad \text{(Einstein's equation)}$$



Albert Einstein
1879-1955

$T_{\alpha\beta} =$ Energy-momentum tensor of matter

$G = 6.6720 \times 10^{-8}\ \text{cm}^3\ \text{g}^{-1}\ \text{sec}^{-2}$ (Gravitational constant)

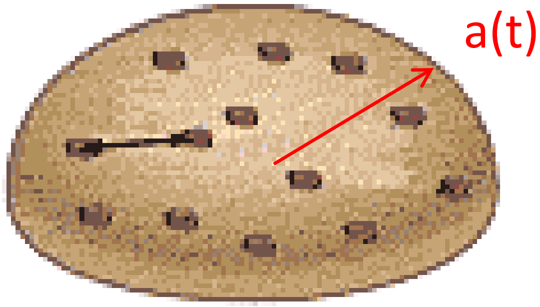
Particle moves on geodesic (shortest distance in the geometry).

Mathematica notebooks:
http://web.physics.ucsb.edu/~gravitybook/
James B. Hartle, *GRAVITY , An Introduction to Einstein's General Relativity* (Addison-Wesley)

The expanding universe

Edwin Hubble

1889 -1953



Hubble's law:

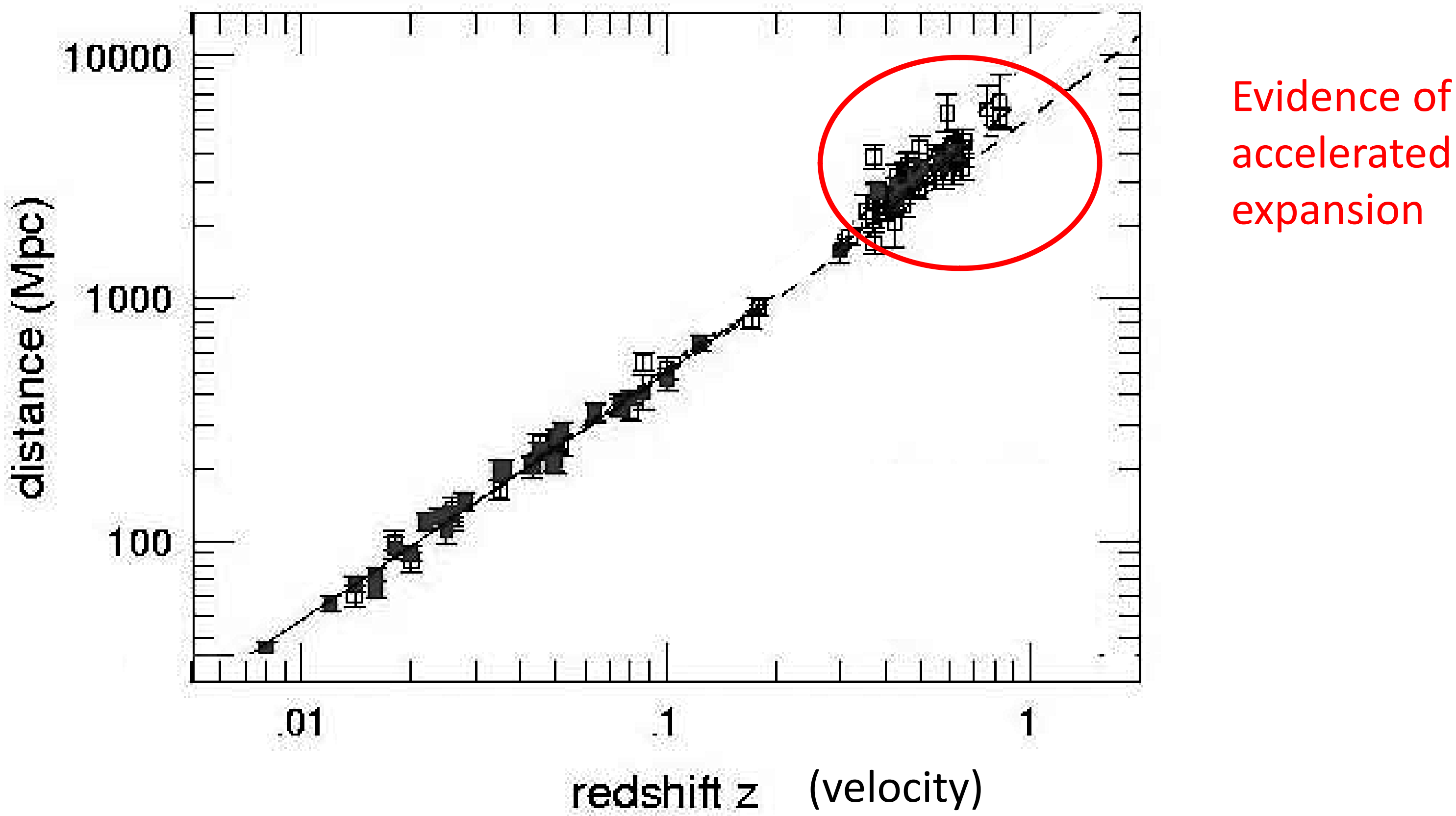
- Distances between galaxies increase with time.
- Rate of increase proportional to distance.
- Extrapolation to distant past: **The big bang**.

Hubble parameter:

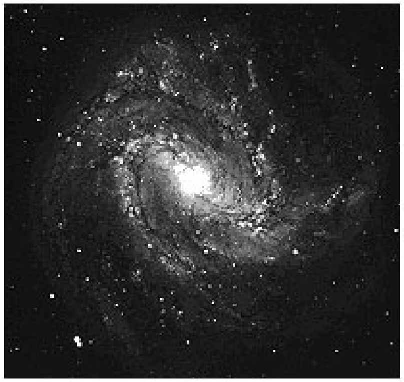
$$H = \frac{1}{a}\frac{da}{dt} = \frac{1}{15\times 10^{9}\ \mathrm{yrs}}$$

Dark energy

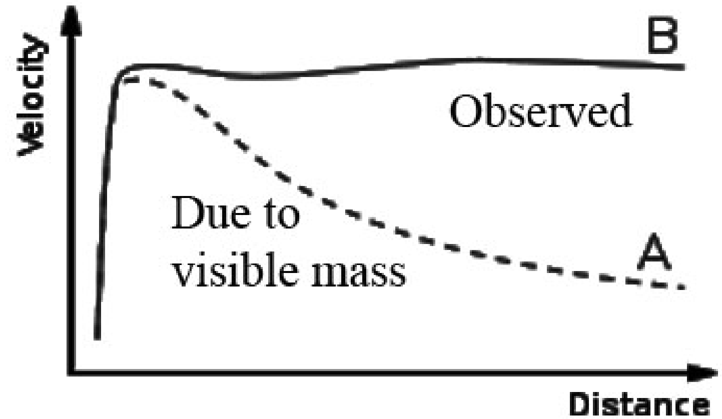
Deviation from Hubble's law:
expansion is accelerating, as if driven by unseen energy.



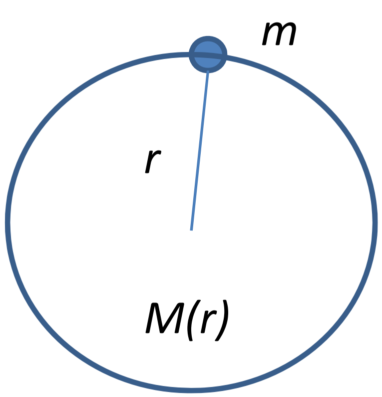
Dark matter: unidentified components of galaxies



Andromeda



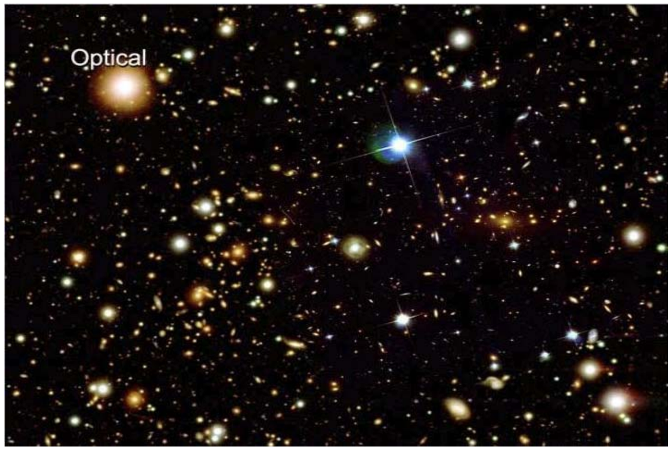
Mass *m* moving in mass distr. *M(r)*



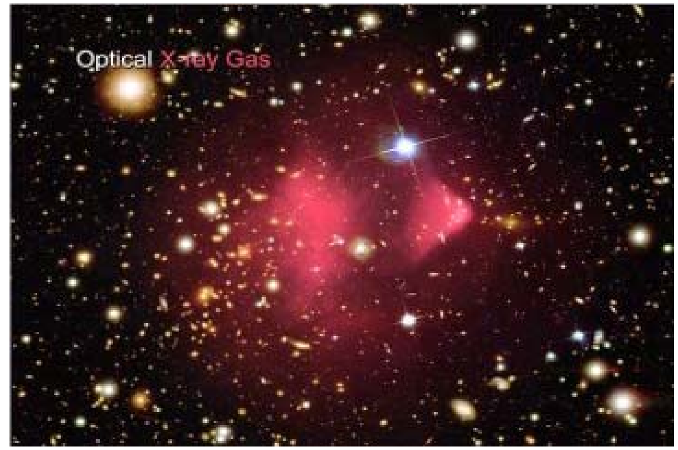
$$\frac{mv^2}{r} = \frac{GmM(r)}{r^2}$$

$$v = \sqrt{\frac{GM(r)}{r}}$$

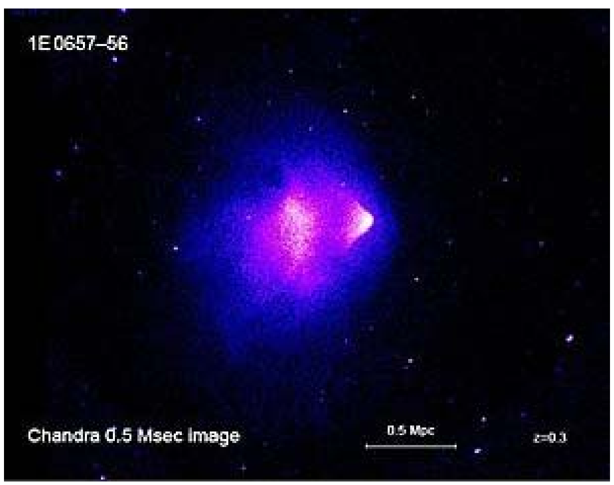
The “bullet cluster”: colliding galaxies



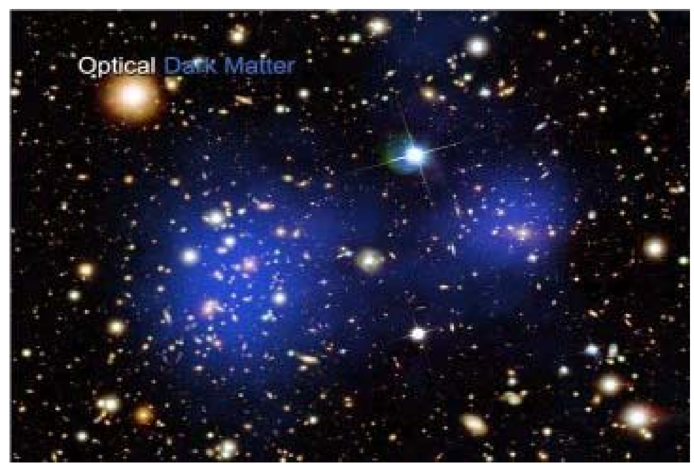
Blue spot: galaxy cluster 1E 0657-56



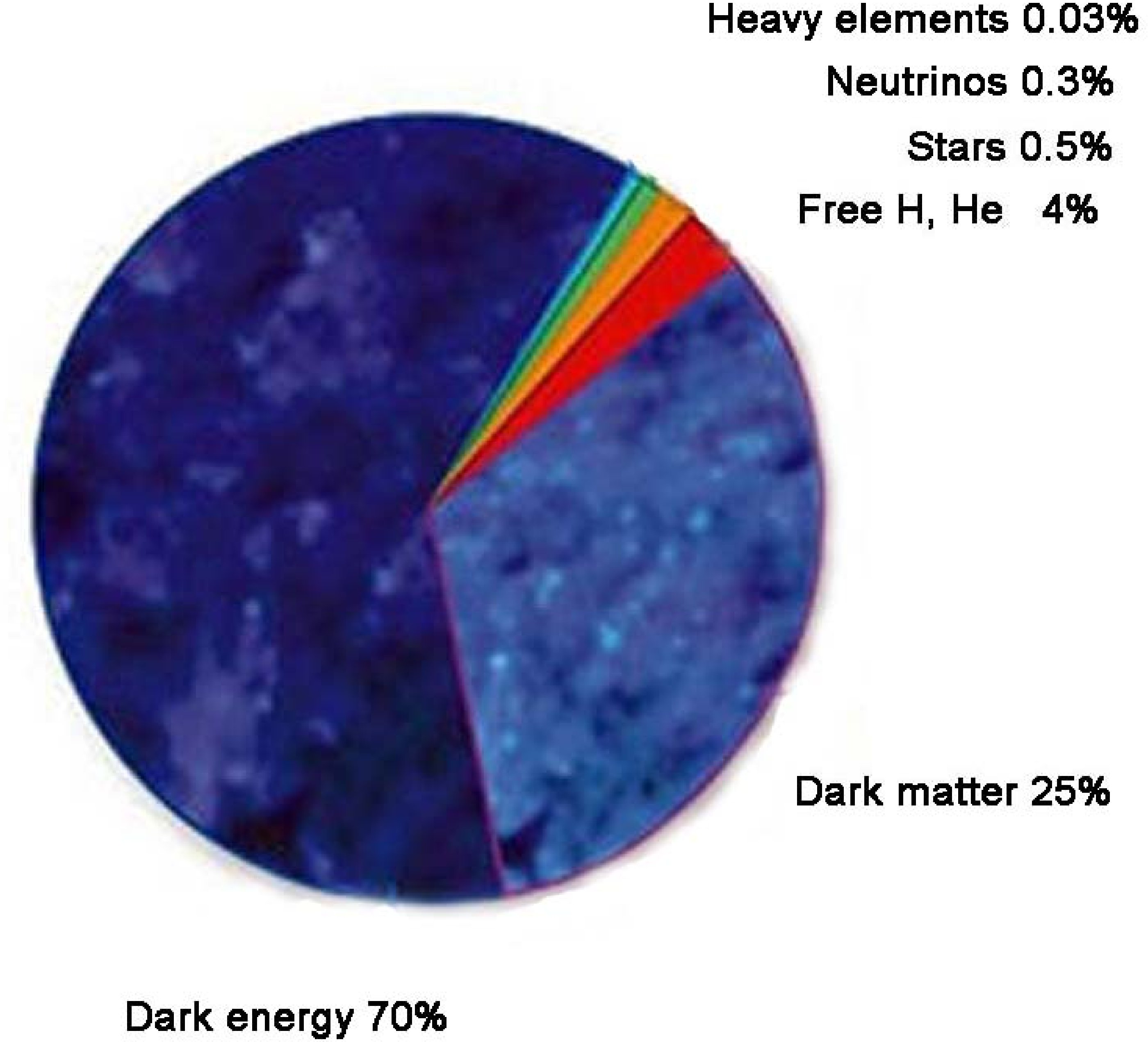
X-ray gas



Overall view



Dark matter


Heavy elements 0.03%
Neutrinos 0.3%
Stars 0.5%
Free H, He 4%
Dark matter 25%
Dark energy 70%

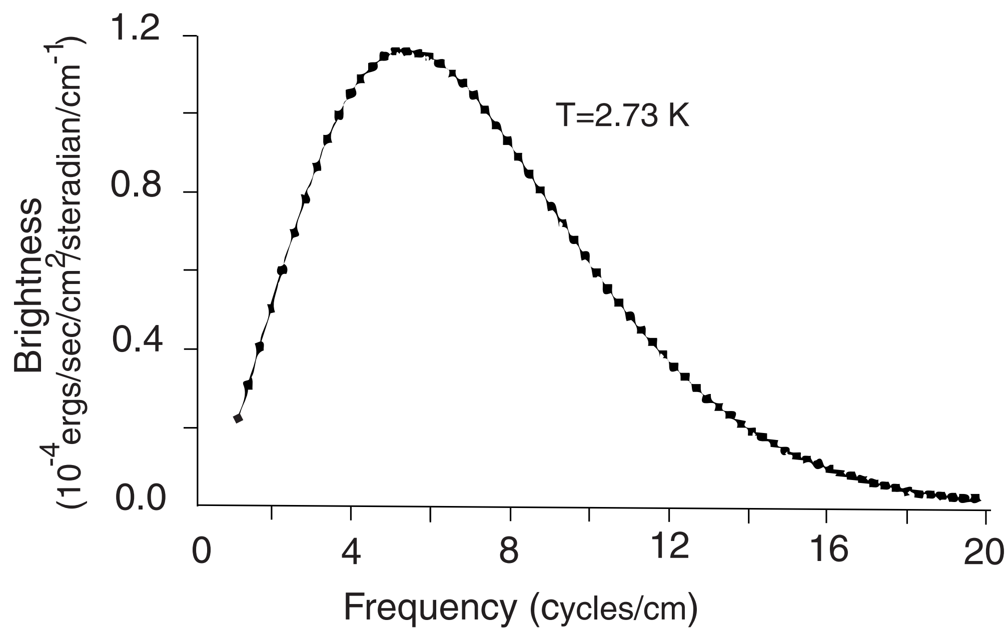
CMB (cosmic microwave background)

- Early universe was a plasma of ionized atoms. Photons were being scattered back and forth among ions and electrons, and cannot propagate.

- At about 10^5 yrs, temperature drops below 10^3 K, and neutral hydrogen was formed. Photons decoupled from matter to become CMB.

- Uniform 3-degree black-body radiation.

- Small angular fluctuations contain info on early universe.



Cosmic inflation

What is the reason for the uniformity on large scale, when different parts of the present universe lie outside of each other's light cone?

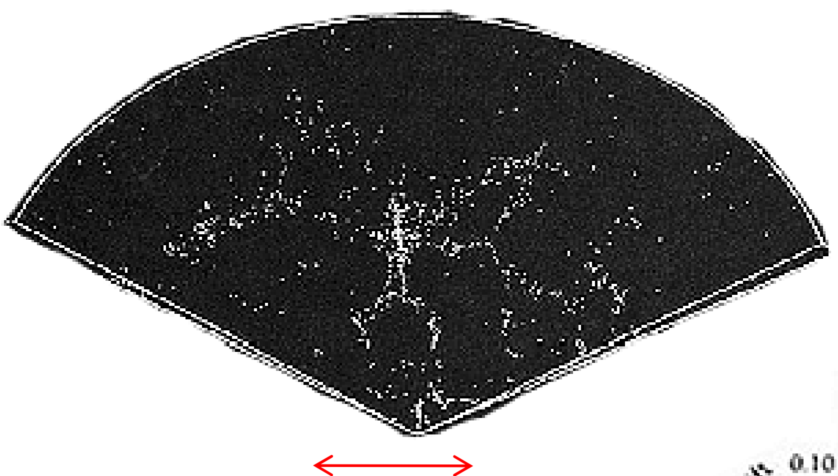
Inflation scenario: matter was uniformly created in a small universe after the big bang. The universe inflates rapidly (by factor 10^{27} in 10^{-26} s), and matter remains uniform.

Models of inflations makes use of scalar field, inspired by the Higgs field in particle theory. In our theory, such a complex scalar field gives rise to superfluity.

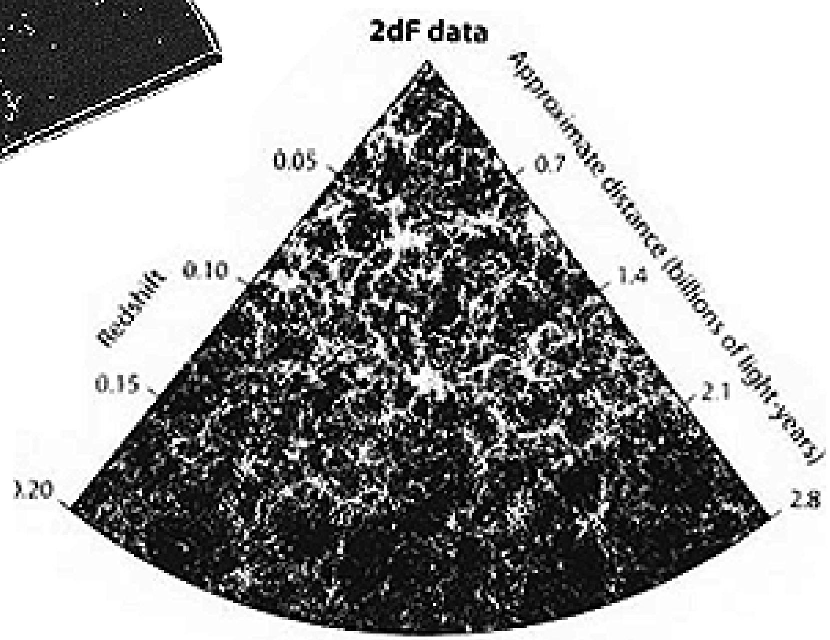
Galactic voids

Though uniform on scale of 10^9 light years, galactic distribution is full of voids on smaller scale.

The "stick man"



10^8 light years



The "Great Wall"

Robertson-Walker metric

Uniform universe, co-moving coordinates.

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$

Curvature parameter:
k = 0, 1, -1

Einstein's equation reduces to

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -\frac{2}{3}T_{00}$$

$$\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]g_{ij} = -2T_{ij}$$

Uniform fluid:

$$T^{00} = -\rho$$

$$T^{ij} = g^{ij}p \quad (i,j = 1,2,3)$$

$$T^{j0} = 0$$

Conservation law ($T^{\mu\nu}_{;\mu} = 0$) :

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + p) = 0$$

FLRW model (Friedmann-Lemaitre-Robertson-Walker)

$$H = \dot{a} / a$$

$$\dot{H} = \frac{k}{a^2} - (p + \rho)$$

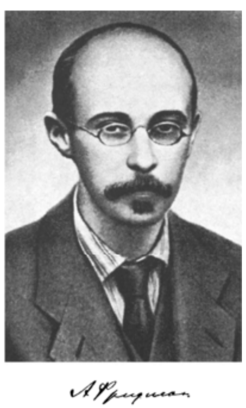
$$H^2 = -\frac{k}{a^2} + \frac{2}{3}\rho$$

$$\dot{\rho} = 3H(\rho + p)$$

Constraint equation

$$1 = -\frac{k}{a^2 H^2} + \frac{2\rho}{3H^2}$$

$$1 = \Omega_k + \Omega_\rho$$



A. Friedmann
(1888-1925)

G. Lemaitre
(1894-1966)

H.P. Robertson
(1903-1961)

A.G. Walker
(1901-2001)

Temperature of CMB has angular dependence across the sky

$$T(\theta) = \sum f_\ell P_\ell(\cos\theta)$$

$$\ell_{\text{peak}} = 200$$

Theory:

$$\ell_{\text{peak}} = \frac{200}{\sqrt{\Omega_\rho}}$$

Thus $\Omega_\rho = 1$

K = 0 (flat universe)

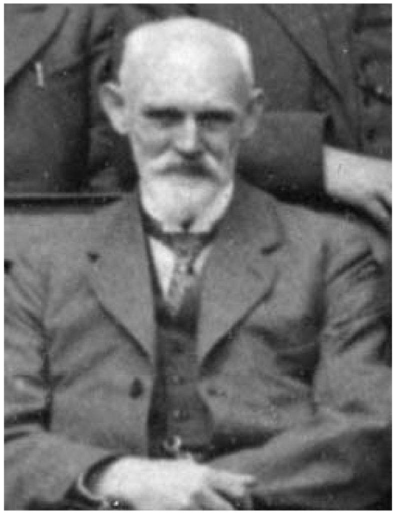
Spherically symmetric metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Substitute into Einstein's equation to find *f(r).*

De Sitter metric: (vacuum solution with cosmological constant)

$$f(r) = 1 - br^2$$



Willem de Sitter
(1872-1934)

This leads to radius = exp *(Ht)*

Accelerated expansion -- dark energy

Fine-tuning problem

Radius of universe = *exp (Ht)*
Hubble parameter: $H = O(1)$ on Planck scale, naturally

$$\text{Planck length} = \sqrt{\frac{\hbar}{c^3} 4\pi G} = 5.73 \times 10^{-35} \text{ m}$$

$$\text{Planck time} = \sqrt{\frac{\hbar}{c^5} 4\pi G} = 1.91 \times 10^{-43} \text{ s}$$

$$\text{Planck energy} = \sqrt{\frac{\hbar c^5}{4\pi G}} = 3.44 \times 10^{18} \text{ GeV}$$

Theory: $H = 10^{43}$ s^{-1}

Observed: H = (Age of universe)$^{-1}$ = (15 billion yrs)$^{-1}$ = 10^{-17} s^{-1}

We would have to "fine-tune" the theory by 60 orders of magnitude!

Schwarzschild metric

$$f(r) = 1 - \frac{2M}{r}$$

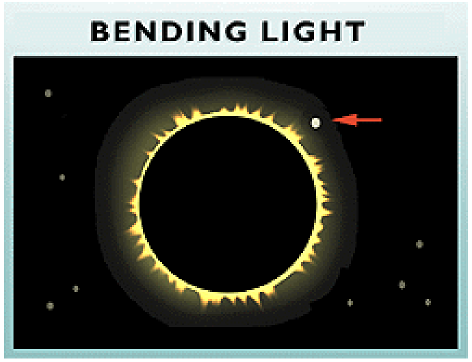
- Vacuum solution (c = G =1)

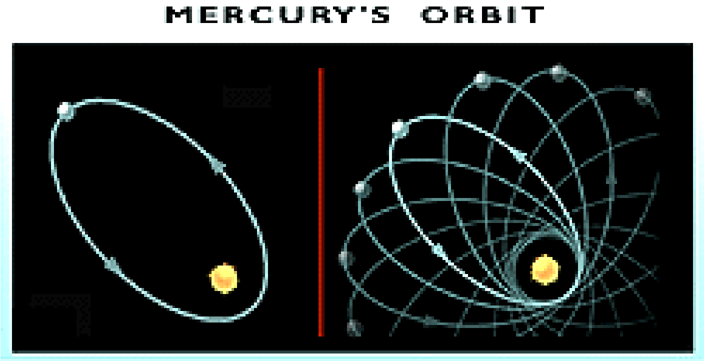
- Reduces to Newtonian gravity at large *r*, with mass *M* at center.

- Schwarschild horizon: r = 2M. Star lying inside horizon will collapse into black hole.

- Corrections to Newtonian gravity:
  - Bending of light by star
  - Precession of perihelion of planetary orbit

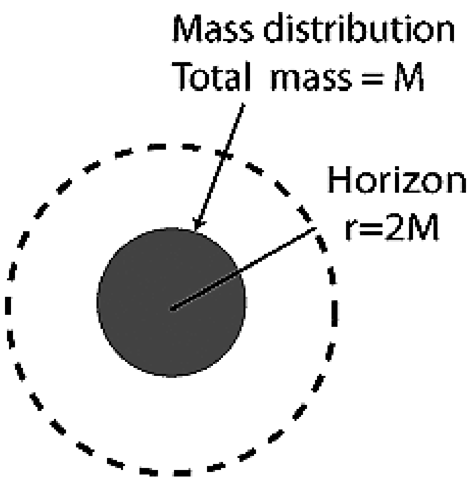
Karl Schwarzschild (1873-1916)



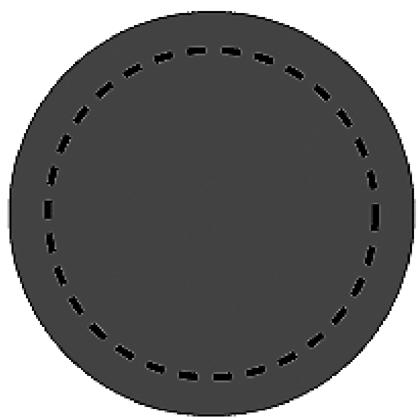


Schwarzchild metric and black hole

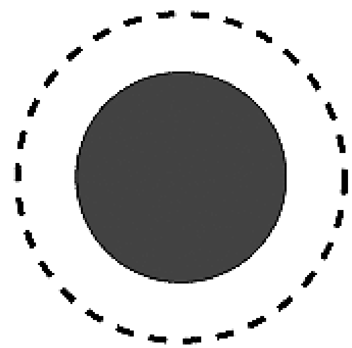
$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$



Schwarzchild metric is valid only outside of mass distribution



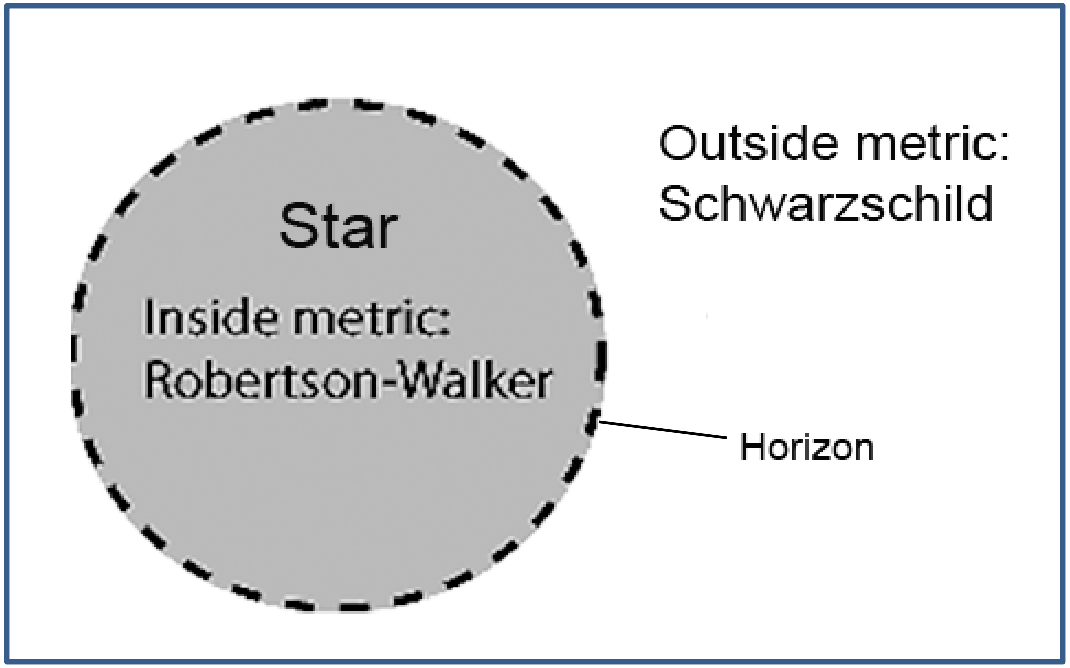
No horizon, no collapse



Will collapse into black hole

Black hole: gravitational collapse

Oppenheimer-Snyder model



Robert Oppenheimer
(1904-1967)

Hartland Snyder
(1913-1962)

- Initial radius = Schwarzschild horizon (R = 2M).
- Solve Einstein's equation for time evolution.
- Join metrics at horizon.

Inside solution

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$ Put *k=1*.

Put pressure *p=0*. Einstein's equation reduces to

$$\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = 0$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = \frac{2}{3}\rho$$

$$\dot{\rho} = -\frac{3\dot{a}}{a}\rho \longrightarrow \rho = c_0 a^{-3}$$

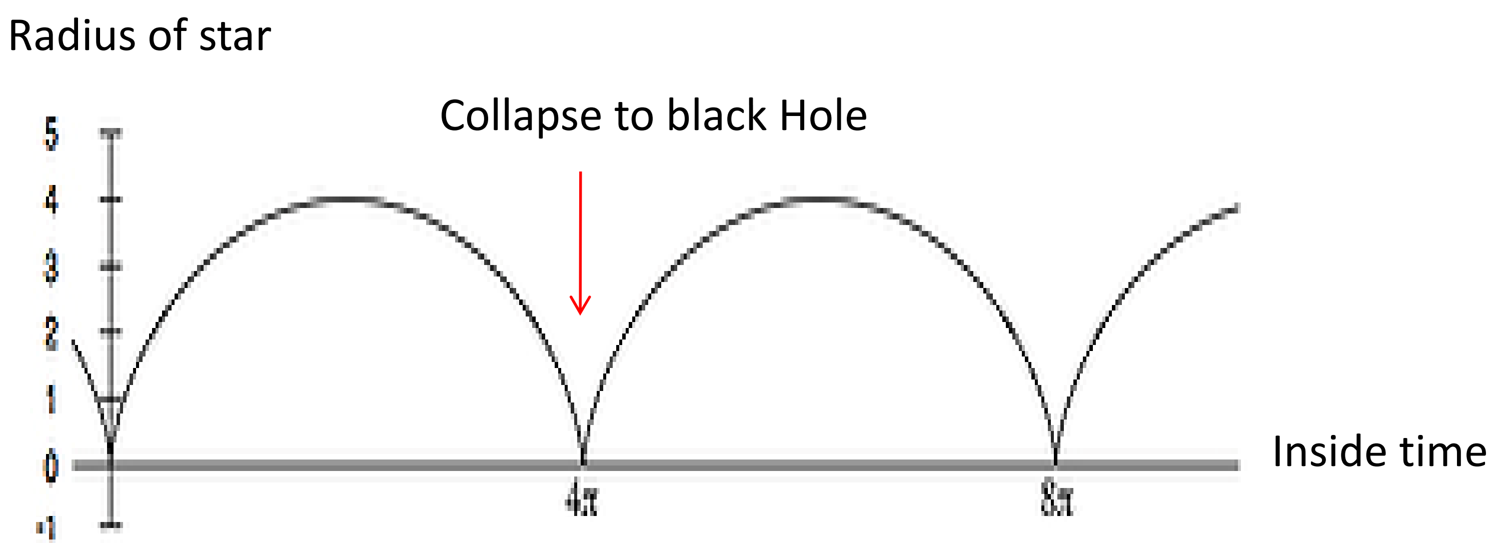
$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = \frac{2c_0}{3a^3}$$

$$\dot{a}^2 = \frac{a(0)}{a} - 1$$

Solution is the cycloid (*a(0)=1*).

$$a = \frac{1}{2}(1 + \cos\psi)$$

$$t = \frac{\psi + \sin\psi}{2\sqrt{k}}$$



- Star radius collapses to zero in finite inside time.
- Joining of metrics gives relation between inside and outside time.
- To an observer outside, the collapse takes infinite time.
- Light emitted from the surface of the star will never reach outside.