A Superfluid Universe

Lecture 4
Research Topics

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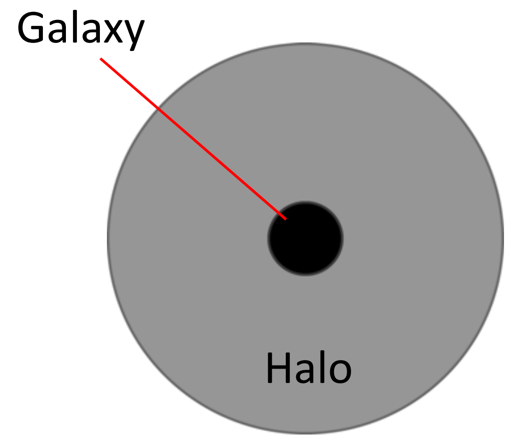
Lecture 4.

- Dark matter
  - Nonlinear Klein-Gordon equation (NLKG)
  - Non-relativistic limit: NLSE
  - Simulation of galactic halo
- Gravitational collapse in a superfluid
  - Kerr metric (rotating star)
  - Frame-dragging -- vorticity from space-time geometry

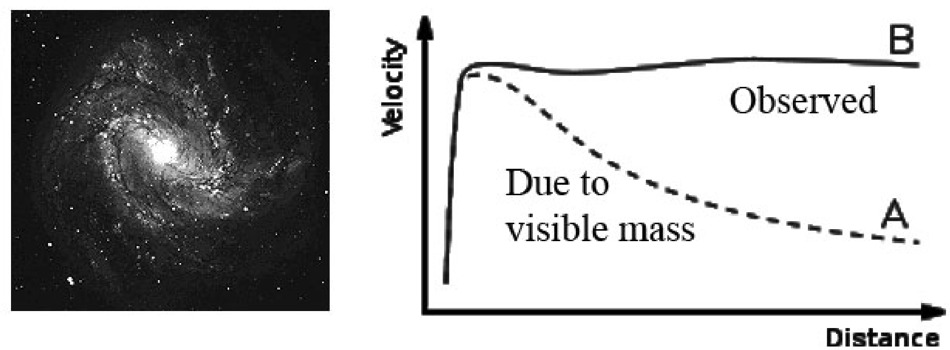
Dark matter



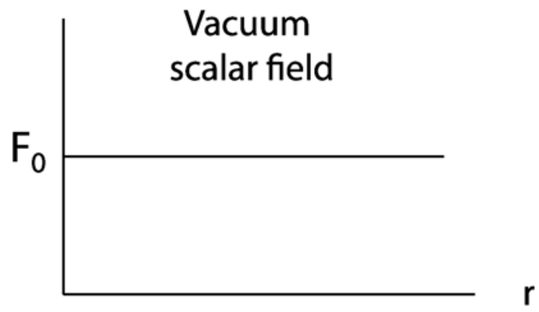
Evidence of dark matter: galaxy halo

- From gravitational lensing
- From velocity curve

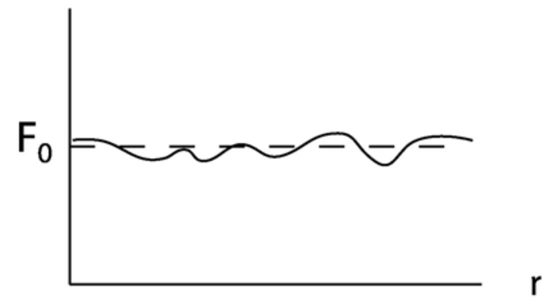
Halo in “bullet cluster” (blue)



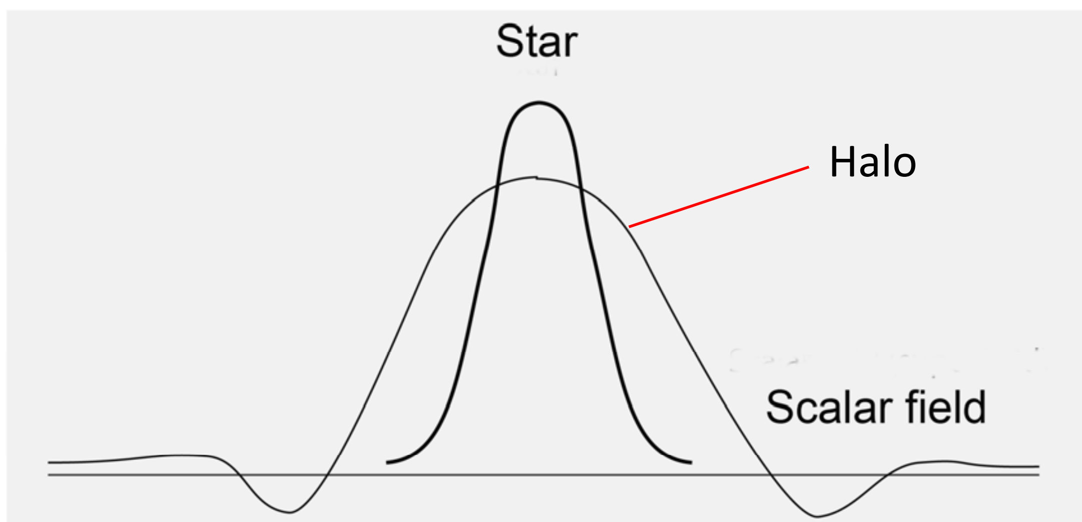
Velocity curve of Andromeda



Energy density = Dark energy



Fluctuations = Dark matter



NLKG (Nonlinear Klein-Gordon equation)

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\phi - \frac{\partial V}{\partial \phi^*} + \frac{\partial \mathcal{L}_{\text{int}}}{\partial \phi^*} = 0 \qquad \phi = F e^{i\sigma}$$

Interaction with "star"

Use any convenient potential, since scale does not change with time, renormalization not important.

$$V = \tfrac{1}{2} g(\phi^* \phi - F_0^2)^2$$

Current density

$$j^\mu = \frac{1}{2i}(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) = F^2 \partial^\mu \sigma$$

$$\partial_\mu j^\mu = 0$$

- This is in flat space-time.
- Gravitational effect of star is regarded as Newtonain, and may be includes in interaction Lagrangian.

The “star”

$$J^\mu = (\rho, \mathbf{J})$$

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

For a rotating star:

$$\mathbf{J} = \rho \mathbf{v}$$

$$v_x = -\Omega y$$

$$v_y = \Omega x$$

The only Lorentz-invariant interaction Lagrangian is a current-current interaction

$$\mathcal{L}_{\text{int}} = -\eta J^\mu j_\mu = -\eta(\mathbf{J} \cdot \mathbf{v} - \rho\dot{\sigma})$$

$$\frac{\partial \mathcal{L}_{\text{int}}}{\partial \phi^*} = -\frac{\eta}{i} J^\mu \partial_\mu \phi = \eta\left(J \cdot \nabla\sigma - \rho\frac{\partial\sigma}{\partial t}\right)$$

A rotating star can create vortices in the superfluid.

$$\left[\nabla^2 - \frac{\partial^2}{\partial t^2} - \gamma\frac{\partial}{\partial t} - g(\phi^*\phi - F_0^2) + \eta\left(J \cdot \nabla\sigma - \rho\frac{\partial\sigma}{\partial t}\right)\right]\phi = 0$$

Damping added by hand

Superfluid velocity

Dimensionless form of NLKG

$$\left[\bar{\nabla}^2 - \frac{\partial^2}{\partial \bar{t}^2} - \bar{\gamma}\frac{\partial}{\partial \bar{t}} - \bar{\phi}^*\bar{\phi} + 1 + \bar{\eta}\left(\bar{\mathbf{J}} \cdot \bar{\boldsymbol{\nabla}}\sigma - \bar{\rho}\frac{\partial \sigma}{\partial \bar{t}}\right)\right]\bar{\phi} = 0$$

Correlation length: $b = \frac{1}{\sqrt{g}\,F_0}$

Relation to physical parameters:

$$x = b\bar{x}$$

$$t = b\bar{t}$$

$$\phi = F_0\bar{\phi}$$

$$\rho = b^{-4}\bar{\rho}$$

$$J = b^{-4}\bar{J}$$

$$\eta = b^3\bar{\eta}$$

$$\gamma = b^{-1}\bar{\gamma}$$

Physical parameters

$$\rho_{\text{dark energy}} \sim 10^{-29}\ \text{g/cm}^3 = 2 \times 10^{-16}\ \text{GeV/cm}^3 = 10^{-2}\ \text{cm}^{-4}$$

Assume scalar field energy supplies dark energy:

$$gF_0^4 = \rho_{\text{dark energy}}$$

$$\sqrt{g}F_0 = \frac{1}{b}$$

$$g = \left(b^4 \rho_{\text{dark energy}}\right)^{-1}$$

Assume correlation length is of galactic size

$$b = 10^{20}\ \text{cm (diameter of Milky Way)}$$

$$g = 10^{-78}$$

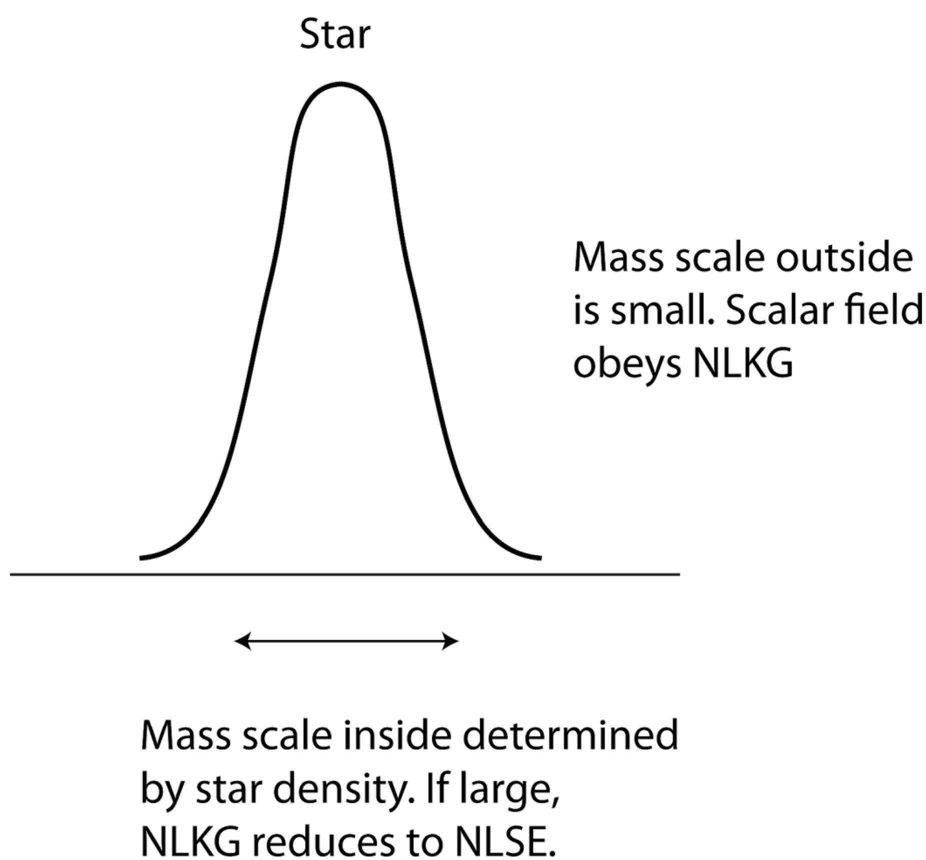
$$F_0 = 10^{19}\ \text{cm}^{-1}$$

$$\tau = \frac{b}{c} = 10^2\ \text{yr}$$

• Length scale in free space is chosen to be of galactic magnitude, corresponding to a small mass scale.

$$m_0 = b^{-1} = 10^{-20}\text{cm}^{-1} = 2 \times 10^{-25}\text{eV}$$

• But inside a star the scale is set by the star density, which may corresponds to a small length, or large mass.



Non-relativistic limit

Relativistic: $E = \pm\sqrt{p^2 + m^2}$ (positive and negative frequencies)

Non-rel: $E = p^2/2\mathrm{m}$ (positive frequency only)

$$\phi = \phi^{(+)} \exp(-im_0 t) + \phi^{(-)} \exp(im_0 t)$$

Put

$$\phi = \psi \exp(-im_0 t)$$

$$\psi = \phi^{(+)} + \phi^{(-)} \exp(-2im_0 t)$$

Second term approaches zero in the weak sense, in the limit $m_0 \to \infty$

$$\left[\nabla^2 - \frac{\partial^2}{\partial t^2} - \gamma\frac{\partial}{\partial t} - g(\phi^*\phi - F_0^2) + \eta\left(J\cdot\nabla\sigma - \rho\frac{\partial\sigma}{\partial t}\right)\right]\phi = 0$$

If star has high mass scale:

$$M = \eta\rho_0 \qquad\qquad M >> b^{-1}$$

Inside star, put

$$\phi = \psi\exp(-iMt)$$

$$\psi = \sqrt{n}\,e^{i\varphi}$$

The "wave function" is still complex

In large M limit the wave function satisfies NLSE:

$$\left[-\frac{1}{2M}\nabla^2 + \lambda(|\psi|^2 - n_0) - \frac{\eta}{2}\left(\rho + \frac{1}{M}J\cdot\nabla\varphi\right)\right]\psi = (i - \gamma')\frac{\partial\psi}{\partial t}$$

$$n_0 = \frac{M}{2}\left(1 + \frac{F_0^2}{M}\right)$$

$$\lambda = \frac{g}{2M}$$

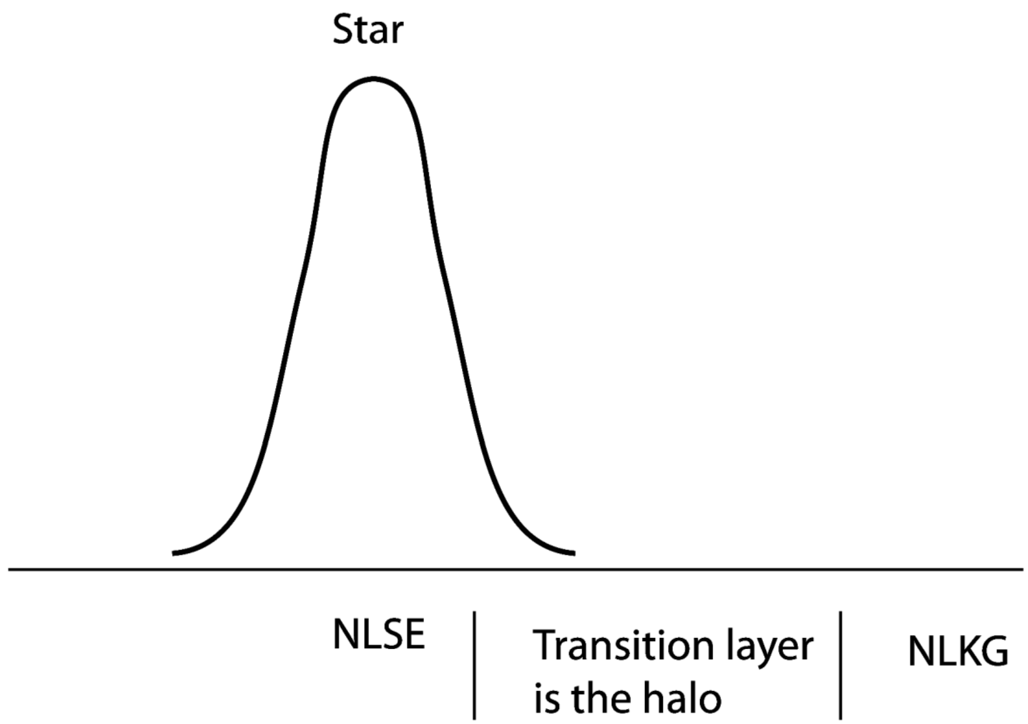
$$\gamma' = \frac{\gamma}{2M}$$

Non-relativistic superfluid velocity:

$$\mathbf{v}_s = \frac{1}{M}\nabla\varphi$$

$$E_{\text{superflow}} = \frac{1}{2}\int d^3x|\psi|^2\mathbf{v}_s^2$$

- Far inside star, superfluid is governed by NLSE.
- Far outside it is governed by NLKG.
- The transition layer is the halo.



The length changes from small to large.
So do vortex core sizes.
Votex density also changes.

Gravitational collapse in a superfluid

- In a Schwarzschild metric, one can repeat the Oppenheimer-Snyder calculation in the presence of a vacuum complex scalar field.

- The result is not qualitatively different.

- To see new effects, and to be more realistic, one has to consider the collapse of a rotating star, which is described by the Kerr metric.

Kerr metric

$$ds^2 = -dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta d\varphi^2$$
$$+ \frac{2mr}{\Sigma}(dt - a\sin^2\theta d\varphi)^2$$

$$a = \frac{J}{m}$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - 2mr + \alpha^2$$

Frame-dragging: the metric has a local angular velocity

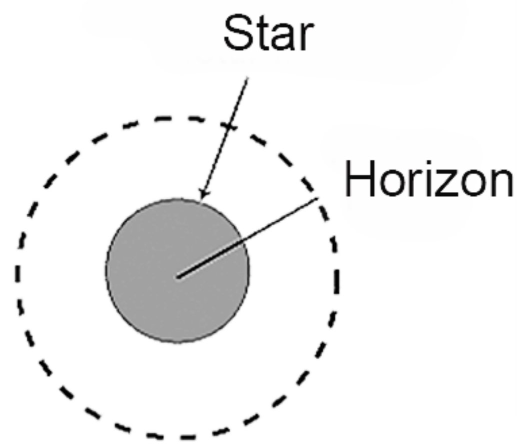
$$\Omega = \frac{a(r^2+a^2-\Delta)}{(r^2+a^2)^2-\Delta a^2\sin^2\theta}$$

• Kerr metric is a solution to the Einstein equation in vacuum.

•Presence of vacuum scalar field will affect the metric.

• One would have to solve Einstein's equation again, with scalar field.

• As first step, one one can study NLKG in Kerr metric



$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right]-\frac{\partial V}{\partial\phi^*}+\frac{\partial\mathcal{L}_{\text{int}}}{\partial\phi^*}=0$$

- In collapse, the length is rapidly changing.

- Must use Halpern-Huang potential.

- Frame-dragging can be transformed away in locally rotating frames.

- This can create vortices.

- After collapse, black hole is formed, but vortices remain outside.

- These could be the “non-thermal filaments, which might constitute “hair” on the black hole.

- A generalization of the Kerr metric is the Kerr-Newmn metric, which describes the condition outside a rotating electrically-charged star.

- Athough there are no charged stars in equilibrium, a star can separate into dipole layers when collapsing.

- This will then describe the inner charges core.

- There would be lightning between layers.