A Productive Asset's Rational Bubble in a Small Open Economy: A Double-Bladed Role of a Credit Constraint

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Abstract

This paper studies a rational price bubble in a productive asset and its effect on the real economy in an overlapping generations model of a small open economy with an analysis on a collateralized credit constraint. As a consequence, the small open economy is vunerable to the bubble emergence. Crucially for the policy-making, the credit constraint plays a double-bladed role in the bubble emergence. That is, with the natural credit limit which is evaluated at the fundamental value of the collateral, a bubble cannot exist; while, if the financial intermediary sets the credit limit at the expected value of the collateral, the credit constraint instead helps support the bubble. Hence, the tight financial regulation and supervision over the credit constraint are recommended for policymakers to prevent or terminate the bubble.

1 Introduction

Throughout history, an small open economy has always been a platform for a price bubble in a productive asset such as land and housing to emerge causing an economic boom and a subsequent painful crash; for example Mexico in early 1980s, Japan in late 1980s[1], and East Asian countries in late 1990s. In common, the large amount of fund is borrowed and invested in the productive asset during the boom period until the price collapses which leads to the widespread default, financial turmoil, and severe recession. In this paper, this phenomenon is explicitly formulated in a format of an overlapping generations model with heterogeneous agents and endogenous supply of the productive asset (referred

[1] Although Japan is normally viewed as a large economy, there is an evidence that in late 1980s the monetary policy seemingly was constrained as the interest rate was kept at the low level despite the overheating economy, see Noguchi[17]. The fixed interest rate characteristic suits the small open economy assumption and results from this paper can be applied. However, the difference of Japan's case from other emerging countries' is that Japan bubble was fueled by its own enormous saving, not from abroad.

to as factory buildings) in both the absence and presence of the collateral credit constraint.

Essentially, three contributions can be highlighted from this paper. Firstly, the small open economy possesses the loose wealth constraint and hence can fuel the bubble with the cheap loan from abroad. Gambling that the bubble will not crash in the near future, agents borrow and purchase bubbly factory buildings in the hope of high capital gain. Secondly, due to the endogenous supply, the bubble induces the over-construction and overutilization of factory buildings which result in an unnecessarily sharp drop in the price level (when the bubble bursts) and the prolonged recession afterward as the economy converges back to the steady state. Lastly and most strikingly, the collateral credit constraint can either terminate or help originate the bubble depending on the type of the credit limit is used: the natural or the expectedly-valued limit.

Particularly, in the stochastic economy with the incomplete market, the credit limit should be set equal to the level known as the natural credit limit (see Aiyagari [2]) which is equal to the lowest possible value of collateralized factory buildings in possession. Since the bubble induces large supply of factory buildings, the fundamental price decreases and hence lowers the ability for agents to afford the bubble. As a result, the rational bubble cannot exist. However, the historical evidence over many financial episodes suggests that the financial sector does not follow the natural credit limit practice: the credit limit rapidly expands according to the market value of factory buildings over the boom and the financial intermediary faces a great loss due to the widespread default in the crash time; for instance, see Dubach and Li [7]. Therefore, the expectedly-valued credit limit is alternatively investigated. With this constraint, as the bubble is growing, the credit limit is being relaxed, and hence helps magnify the bubble in the positive feedback loop fashion. Furthermore, the expectedly-valued can even help endogenise the initial bubble[2]. In regard of the policy implication, this double-bladed role of credit constraint calls for the careful financial regulation and supervision to prevent the bubble.

Several works are related to this paper. In the general equilibrium setting, the transversality condition is the main reason for the non-existence of bubbles (for example, see Tirole [22], Obstfeld and Rogoff [18], Magill and Quinzii [16], Santos and Woodford [20], and Kocherlakota [12]). However, Tirole [23] and Caballero and Krishnamurthy [3] shows that in the overlapping generations framework bubbles can emerge if the economy is dynamically inefficient. In this paper, relaxing the transversality condition by using the small open assumption with the definite crash probability setting is adopted rather than the dynamically inefficiency argument for the bubble to emerge.

The most related framework to the present paper is by Weil [24] which incorporates the probability for the price of an asset to once-and-for-all collapse to its fundamental value. Additionally, the collateralized credit constraint in

[2] As in Diba and Grossman [5, 6], Weil [25], and Jarrow, Protter, and Shimboposits [10], if a rational asset bubble exists, it must have started on the first day of trading. This present paper also shares this feature. The expectedly-valued credit constraint can help endogenise the bubble in the first period with respect to the unexpected shock on the interest rate.

the spirit of Kiyotaki and Moore [11], where an asset (factory buildings in this paper) play the dual role as being a factor of production and the collateral for the loan, is introduced. The presence of the collateral credit constraint brings about the balance-sheet effect which is studied extensively in the literature (for instance, see Krugman [14], Aghion, Bacchetta, and Banerjee [1], and Schneider and Tornell [21]). Although the effect disappears in the natural credit limit, it is crucial in the expectedly-valued credit limit case.

This paper is organized as follows. Section 2 analyzes the deterministic economy after the bubble crash. Based on the fundamental price level derived in the after-crash economy, Section 3 examines the stochastic before-crash economy in various environments. Next, the effect of the bubble on the real output is provided in Section 4. Lastly, Section 5 concludes the paper.

2 After-Crash Economy

This section firstly examines the deterministic economy to search for the fundamental price path so that the next section can study a bubble in the stochastic economy defined upon this fundamental price by using backward induction. Consider an overlapping generations model of a small open economy with two-period-lived agents and the perfect international capital mobility. The economy faces the fixed world interest rate $r^* \in R_+$ and all markets are competitive.

There are two goods in this economy which are a factory building [3] and a composite-consumption good. The price of the consumption good is set equal to 1 as the numéraire. Denote $p_t \in R_{++}$ as the period t price of factory buildings in term of consumption good. In addition, factory buildings are the non-tradable good.

Each generation is populated with n_1 contractors and n_2 producers. Each type is initially endowed with the constant amount of consumption good W. At the end of the young period, each contractor and producer accesses to technology to construct factory buildings and to produce consumption good respectively. Available saving channels are only through the financial intermediary and available production.

Denote subscript $i = 1, 2$ referring to variables related to the contractor and to the producer correspondingly. Both types have a linear life-time utility function $c_{i1t} + \beta c_{i2t}$, where $c_{i1t}, c_{i2t} \in R_+$ represent the consumption of a generation t agent of type i when they are young and old respectively, and $\beta \in (0, 1]$ is a discount factor[4]. Additionally, the patient economy where $\beta(1 + r^*) > 1$ is assumed[5]

[3] It can be interpreted as any durable good which is an input of the production of goods and services, for example working offices and kiosks.

[4] Note that the model specification is designed for the tractability reason. In order to be capable of later studying the credit constraint analytically, the risk averse preference and the wage dynamic are drop off.

[5] This patient economy assumption is not crucial. The analysis over the impatient economy where $\beta(1 + r^*) < 1$ can also be conducted and similar insights will be obtained. Since considering this case costly lengthens the paper, it is ignored here.

The young contractor chooses the amount of the borrowing $b_{1t} \in R$ and investment in capital input $k_t \in R_+$ which is used to construct new factory buildings in the next period $y_{1t} \in R_+$ through the concave production function assumed to be of the Cobb-Douglas form: $y_{1t} = Ak_t^{\alpha}$. The capital is depreciated at rate $\delta \in (0, 1]$.

Given the factory depreciation rate $\theta \in (0, 1]$, the no-arbitrage condition between the borrowing and capital investment results below.

$$r^* + \delta = p_{t+1}\alpha Ak_t^{\alpha-1} \tag{1}$$

Intuitively, the contractor's demand for capital input depends on tomorrow's price of factory output: the higher factory price prevails next period, the more profitable the contractor perceives and the more he invests in factory buildings today.

Next, according to Equation 1, the law of motion of factory stock per producer can be written below: tomorrow's level of factory stock is the sum between today's level of factory stock and the newly-built factory buildings, which are determined by tomorrow's factory price.

$$f_{t+1} = (1-\theta)f_t + \Gamma p_{t+1}^{\frac{\alpha}{1-\alpha}} \tag{2}$$

where $\Gamma = \frac{An_1}{n_2}\left(\frac{\alpha A}{\delta+r^*}\right)^{\frac{\alpha}{1-\alpha}}$.

For the young producer, the decision between borrowing $b_{2t} \in R$ and factory purchase $f_t \in R_+$, which is used to produce consumption good in the next period $y_{2t} \in R_+$ through the concave production function assumed to be of the Cobb-Douglas form Bf_t^{ε}, is processed. Additionally, he faces the following credit constraint.

$$(1+r^*)b_{2t} \leq (1-\theta)p_{ct+1}f_t \tag{3}$$

where p_{ct+1} is the price which the financial intermediary uses to evaluate the value of the collateral next period.

This credit constraint limits the amount of borrowing based on value of the collateral which is factory buildings in possession. In the spirit of Kiyotaki and Moore [11], this happens because of the asymmetric information problem. The risk-neutral financial intermediary cannot verify how much consumption good the producer actually produces. Thus, the producer may intentionally misreport the productivity and default. On the contrary, factory buildings cannot be secretly sold due to the durability and the immovability which make the verification on contractors relatively easy. As a result, to eliminate the default risk on the part of producers, factory buildings are required as the collateral for the loan. In other words, the debt obligation cannot exceed the future value of factory buildings in possession.

For the sake of analysis, firstly consider the baseline case where there is no credit constraint for the producer. Then, use the obtained result to analyze the economy with the credit constraint afterward.

2.1 Without credit constraint: the baseline model

In the absence of credit constraint, the producer demands for factory buildings optimally at the level that equates the return of factory investment to the risk-free return as in Equation 4.

$$\frac{\varepsilon B f_t^{\varepsilon-1} + (1-\theta)\, p_{t+1}}{p_t} = (1+r^*) \tag{4}$$

Therefore, according to Equation 2 and Equation 4, the after-crash system in the baseline case is presented below as the system of two first-order difference equations. Equation 5 provides the positive steady state of the system. Denote $x_t = (p_t, f_t)$, $\bar{x} = (\bar{p}, \bar{f})$, and define the after-crash system below as $x_{t+1} = \phi(x_t)$ where $\phi : R_+^2 \to R_+^2$.

$$\left[\begin{array}{c} p_{t+1} \\ f_{t+1} \end{array}\right] = \phi(x_t) = \left[\begin{array}{l} \left\{\begin{array}{lll} \left(\frac{1+r^*}{1-\theta}\right) p_t - \frac{\Omega}{(1-\theta) f_t^{\Pi}} & \text{if} & \text{positive} \\ 0 & \text{if} & \text{otherwise} \end{array}\right. \\ (1-\theta) f_t + \Gamma p_{t+1}^{\Psi} \end{array}\right]$$

$$\left[\begin{array}{c} \bar{p} \\ \bar{f} \end{array}\right] = \left[\begin{array}{c} \left(\frac{\theta}{\Gamma}\right)^{\frac{\Pi}{1-\Pi\Psi}} \left(\frac{\Omega}{r^*+\theta}\right)^{\frac{1}{1+\Pi\Psi}} \\ \left(\frac{\Gamma}{\theta}\right)^{\frac{\Pi}{1-\Pi\Psi}} \left(\frac{\Omega}{r^*+\theta}\right)^{\frac{\Psi}{1+\Pi\Psi}} \end{array}\right] \tag{5}$$

where $\Pi = 1-\varepsilon$, $\Psi = \frac{\alpha}{1-\alpha}$, and $\Omega = \varepsilon B$[6].

The after-crash dynamics are defined by sequences of factory price and stock $\{p_t\}_{t=T}^{\infty}$ and $\{f_t\}_{t=T}^{\infty}$ where T is the time of crash such that they satisfy the system ϕ with a given factory stock of the crash period f_T. The system ϕ demonstrates the rich interaction between today's and tomorrow's levels of both factory price and stock. The supply of newly-built factory buildings from the contractor depends on tomorrow's factory price. The demand for factory buildings depends not only on the future price in term of capital gain, but also on the factory stock itself in term of rent. Given today's price, if today's factory stock is large, which translates into the low rent, tomorrow's price has to be high for the no-arbitrage condition (Equation 4) to hold.

Standard definition of the fundamental value of an asset (for example, see Santos and Woodford [20]) is the expected present value of the stream of its dividends. This definition becomes problematic in this context. To see this, Equation 4 can be re-written as follows.

$$p_T = \frac{l_{T+1}}{1+r^*} + \frac{(1-\theta) l_{T+2}}{(1+r^*)^2} + \frac{(1-\theta)^2 l_{T+3}}{(1+r^*)^3} + \ldots$$

where l_t is denoted as the rent the producer attain at period t: $l_t = \varepsilon B f_t^{\varepsilon-1}$.

[6] The lower-bound condition in the system ϕ implies that when the expectation of the tomorrow's factory price is negative, factory buildings become useless and have zero value. Consequently, no new factory building will be produced thereafter.

Letting $\lim_{t\to\infty} \frac{(1-\theta)^{t-1} l_{T+t}}{(1+r^*)^t} = 0$ normally implies no bubble in the standard definition. However, the system ϕ implies that the sequence of rents $\{l_t\}_{t=T}^{\infty}$ in turn relies on p_T itself. In other words, different p_T generates different corresponding sequence of rents so any p_T can be all considered as the fundamental price in the standard definition. With this price-dividend dependency dissatisfaction, new definition is demanded.

Definition 1 A function $\rho(f_t)$ where $\rho : R_{++} \to R_{++}$ is a fundamental price function if for any $f_t \in R_{++}^2$, $(\rho(f_{t+1}), f_{t+1}) = \phi(\rho(f_t), f_t)$ and $\phi^{\{n\}}(\rho(f_T), f_T)$ is finite for all $n \in N$ where $\phi^{\{n\}}$ means n-time iteration of the system ϕ. ■

In words, the fundamental price here is the positive equilibrium price path whose price and stock both do not explode eventually. This resembles the common transversality condition or wealth constraint (see Tirole [23] and Obstfeld and Rogoff [18]). Intuitively, an open economy within the world of finite number of countries can still afford the factory stock with an increasing price only if the growth rate of the world is higher than the growth rate of the bubble[7]. Otherwise, there will be a generation that cannot afford factory buildings in the future which causes the solution infeasible in the forward-looking model. However, the small open economy assumption together with the exogenous crash probability generates the possibility for the finite-period explosive price path.

Now, we are in the position to define a bubble used in this paper.

Definition 2 A bubble is a difference between the actual and the fundamental price defined in Definition 1: $p_t - \rho(f_t)$. ■

Thereby, the definition of a bubble here is weaker than the standard one due to the stronger definition of the fundamental value. Since the price of the asset can affect its future dividends, the bubble may be easily misperceived as the fundamental price. This is probably the reason why bubbles in the real world are very hard to be early detected, not until it bursts.

From Definition 2, the fundamental price function is required in order to study a behavior of the bubble. The below proposition serves this purpose. In words, the proposition states that there exists the unique fundamental value at any given level of factory stock. Moreover, the more factory stock in the economy, the lower the fundamental price becomes. Intuitively, when supply of factory buildings rises, the price decreases to clear the market, see Figure 1.

Proposition 1 For the system ϕ, there exists $\rho(f_t)$ which is unique, continuous, strictly decreasing, and satisfying $\lim_{n\to\infty} \phi^{\{n\}}(\rho(f_T), f_T) = \bar{x}$. ■

Proof Basically, $\rho(f_t)$ is derived from the globally stable manifold of ϕ. See Appendix. ■

[7] Note that the zero population and endowment growth assumption implies the zero growth rate of the world.

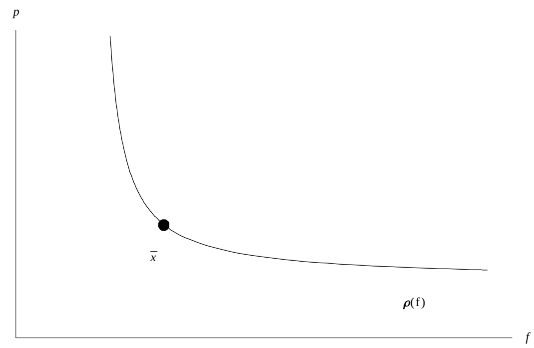

Figure 1: The fundamental price path

2.2 With credit constraint

In the deterministic context, p_{ct+1}in Equation 3 is simply equal to the actual future price p_{t+1}. Now, the producer's desired level of borrowing b_{2t} may not be granted. Since the producer is patient $\beta(1+r^*) > 1$, he consumes only when he is old $c_{21t} = 0$, so the optimal debt obligation is expressed below.

$$(1 + r^*)\, b_{2t} = (1 + r^*)\, (p_t f_t - W)$$

From Equation 4, the collateral value can be written as $(1 - \theta)p_{t+1}f_t = (1 + r^*)p_t f_t + \varepsilon B f_t^{\varepsilon}$. Consequently, the critical value $\hat{f}$ can be defined below as the level of factory stock that equates the debt obligation to the collateral value.

$$\hat{f} = \left[\frac{(1 + r^*)W}{\varepsilon B}\right]^{\frac{1}{\varepsilon}}$$

Hence, for any $f_t \geq \hat{f}$ the credit constraint binds and vice versa. Note that $\frac{\partial \hat{f}}{\partial W} > 0$. Intuitively, when the producer possesses the larger amount of endowment, he borrows less and the economy tends to be less credit-constrained.

Next, the binding credit constraint case, using $(1+r^*)b_{2t} = (1-\theta)p_{t+1}f_t$ as a constraint, is analyzed. This applies only to the economy with $f_t \geq \hat{f}$. The factory investment optimal condition is derived below.

$$\frac{\beta(1 + r^*)\varepsilon B f_t^{\varepsilon-1} + (1 - \theta)\, p_{t+1}}{p_t} = (1 + r^*) \tag{6}$$

Considering this case leads to the possibility of violating the young consumption non-negativity constraint $c_{21t} \geq 0$. From Equation 6 and the binding credit constraint, the optimal debt can be derived as follows.

$$b_{2t} = p_t f_t - \beta \varepsilon B f_t^{\varepsilon}$$

Then, the other critical value $\tilde{f}$ can be defined as the level of factory stock that makes $c_{21t} = 0$.

$$\tilde{f} = \left[\frac{W}{\beta\varepsilon B}\right]^{\frac{1}{\varepsilon}}$$

Thereby, conditional on $f_t \geq \hat{f}$, for any $f_t \geq \tilde{f}$ the non-negativity constraint binds and vice versa, and the optimal factory investment condition is derived from the binding credit constraint and $c_{21t} = 0$, resulting in Equation 7 below.

$$\frac{(1+r^*)Wf_t^{-1} + (1-\theta)\,p_{t+1}}{p_t} = (1+r^*) \tag{7}$$

Since $\beta(1+r^*) > 1$ is assumed, $\tilde{f} < \hat{f}$; thus, the domain of factory stock $f \in R_{++}$ is divided into two regimes only: $f \in (0, \hat{f}]$ is the non-binding regime (*non*) where Equation 4 holds, and $f \in [\hat{f}, \infty)$ is the credit-binding regime (*cb*) where Equation 7 holds[8].

Denote $\bar{x}_j = \left(\bar{p}_j, \bar{f}_j\right)$, and define the below system as $x_{t+1} = \phi_j(x_t)$ where $\phi_j : R_+^2 \to R_+^2$ and $j = non, cb$.

$$\phi_j(x_t) = \left[\begin{array}{c} p_{t+1} \\ f_{t+1} \end{array}\right] = \left[\begin{array}{l} \begin{cases} \left(\frac{1+r^*}{1-\theta}\right)p_t - \frac{\Omega_{jt}}{(1-\theta)f_t^{\Pi}} & \text{if positive} \\ 0 & \text{if otherwise} \end{cases} \\ (1-\theta)f_t + \Gamma p_{t+1}^{\Psi} \end{array}\right]$$

$$\left[\begin{array}{c} \bar{p}_j \\ \bar{f}_j \end{array}\right] = \left[\begin{array}{c} \left(\frac{\theta}{\Gamma}\right)^{\frac{\Pi}{1-\Pi\Psi}}\left(\frac{\bar{\Omega}_{jt}}{r^*+\theta}\right)^{\frac{1}{1+\Pi\Psi}} \\ \left(\frac{\Gamma}{\theta}\right)^{\frac{\Pi}{1-\Pi\Psi}}\left(\frac{\bar{\Omega}_{j}}{r^*+\theta}\right)^{\frac{\Psi}{1+\Pi\Psi}} \end{array}\right] \tag{8}$$

where $\Omega_{non,t} = \bar{\Omega}_{non} = \varepsilon B$ for the non-binding system, and $\Omega_{cb,t} = \frac{(1+r^*)W}{f_t^{\varepsilon}}$ with $\bar{\Omega}_{cb} = \frac{(1+r^*)W}{f_{cb}^{\varepsilon}}$ for the credit-binding system[9].

Proposition 2 Given $\beta(1+r^*) > 1$, there exists a time-invariant critical value $\hat{f} = \left[\frac{(1+r^*)W}{\varepsilon B}\right]^{\frac{1}{\varepsilon}}$ which separates the system into the non-binding regime $f < \hat{f}$ and the credit-binding regime $f > \hat{f}$. ■

Proof As already shown in the text. ■

To prevent any confusion, the following terminology is defined.

Definition 3 The system ϕ_j is regime-dependent if j is changed according to which regime the dynamics currently are located in[10]. ■

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[8] Note that the credit-binding regime here is defined as both the credit and non-negativity constraints are binding.

[9] Note that the system ϕ in Proposition 1 is exactly the same as the system ϕ_{non}.

[10] For example, suppose that (p_T, f_T) is located in the regime *non* and then jumps into the regime *cb* in the next period. Then the dynamics in the period $T+2$ are determined as follows: $(\rho_{T+2}, f_{T+2}) = \phi_{cb}(\phi_{non}(p_T, f_T))$.

Hence, the after-crash dynamics are defined by sequences of factory price and stock $\{p_t\}_{t=T}^{\infty}$ and $\{f_t\}_{t=T}^{\infty}$ where T is the time of crash such that they satisfy the regime-dependent ϕ_j with a given factory stock of the crash period f_T. The steady state in Equation 8 and the critical values $\hat{f}$ in Proposition 2 divide an analysis into two different cases which depends upon the level of the initial endowment. To be precise, this low or high endowment translates into whether the valid steady state of factory stock is located on the right or the left of $\hat{f}$ respectively. For a given set of parameter values, the economy can only fall into one of the following two cases[11].

- Economy 1: $\bar{f}_{non} < \bar{f}_{cb} < \hat{f}$.
- Economy 2: $\hat{f} \leq \bar{f}_{cb} < \bar{f}_{non}$.

As in the baseline model, the deterministic after-crash economy is the underlying ground for the bubble in the stochastic before-crash economy to build on; hence, the fundamental price path is firstly needed to determined. The fundamental price in Definition 4 below is a slight modification of Definition 1 to suit the more complex system with credit constraint.

Definition 4 A function $\rho(f_t)$ where $\rho : R_{++} \to R_{++}$ is a fundamental price function if for $f_t \in R_{++}$, $(p(f_{t+1}), f_{t+1}) = \phi_j(\rho(f_t), f_t)$ and $\phi_j^{\{n\}}(\rho(f_T), f_T)$ is finite for all $n \in N$, where ϕ_j is regime-dependent. ■

Proposition 3 below characterizes the fundamental price function in each of the above two economies. It turns out that the fundamental price with the same topological properties as in Proposition 1 is obtained: unique, continuous, and strictly decreasing in f.

Proposition 3 Given $f_T \in R_{++}$ and regime-dependent ϕ_j, there exists a unique, continuous, and strictly decreasing fundamental price function $\rho : R_{++} \to R_{++}$ in which,

- for Economy 1: $\lim_{n\to\infty} \phi_j^{\{n\}}(\rho(f_T).f_T) = \bar{x}_{non}$.
- for Economy 2: $\lim_{n\to\infty} \phi_j^{\{n\}}(\rho(f_T).f_T) = \bar{x}_{cb}$. ■

Proof See Appendix. ■

Intuitively, being constrained by the credit constraint and non-negativity constraint should result in the lower fundamental value than it would have been without the constraint. In particular, agents need a sufficient borrowing to purchase factory buildings at the normal fundamental value. Credit constraint limits the ability to afford factory buildings at the normal fundamental value; consequently, the fundamental value is required to fall. Proposition 4 captures this intuition.

Proposition 4 The binding credit and non-negativity constraints reduce the level of the fundamental value. ■

[11] Note that in the view of Proposition 2 there is only one valid steady state in each case . For example, Economy 1's steady state is $\bar{f}_{non}$, not $\bar{f}_{cb}$. $\bar{f}_{cb}$ is stated to show the topological property of the system and it is useful in the proof of Proposition 3 below.

Proof This is proved by directly comparing the fundamental price function derived from Proposition 1 with the one from Proposition 3, see Figure 2a-b[12] ■

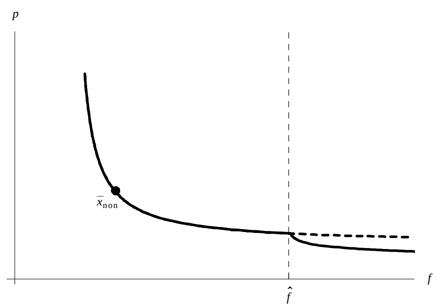

Figure 2a: $\rho(f)$ of Economy 1

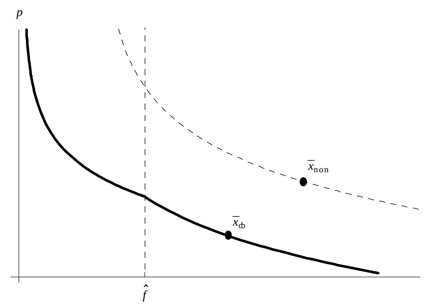

Figure 2b: $\rho(f)$ of Economy 2

3 Before-crash economy

Based on the deterministic economy in the former section, now the stochastic economy is analyzed. Taking the after-crash economy as given, the economy follows a Markov process between two states: optimism and pessimism. Being optimistic today, there is a fixed probability q to change to the pessimism next period. It is assumed that the pessimism occurs once and for all. The transition matrix is given below.

$$\begin{bmatrix} 1-q & 0 \\ q & 1 \end{bmatrix}$$

This Markov process certifies that in the future the optimism will crash with probability one. This is easily seen as shown below.

$$\Pr(E) = \lim_{t\to\infty} (1-q)^t = 0$$

where Pr means probability and E is an event that the pessimism never occurs for all time[13].

As long as agents are still confident about the boom period and enjoying the high capital gain financed by the overseas capital inflow, the economy can be off the fundamental path temporarily. In particular, the before-crash economy operates by means of the expected value between optimistic and pessimistic states. The pessimistic state is referred to as the after-crash economy which has

[12] The solid line is the fundamental price function when there is the credit constraint; while, the dash line is the one when there is no credit constraint.

[13] The argument that agents expect the pessimism to eventually occur and the bubble will surely collapse so the borrowing from abroad will not reach infinity is debatable. One can address that the expected future borrowing still can become very large and this may violate the small open economy assumption. However, this is the best justification that permits examination of a bubble-followed-by-crash scenario.

been analyzed. Therefore, the optimistic price also endogenously rests on the fundamental price. When agents suddenly become pessimistic, the factory price sharply collapses to the fundamental price. Denote p_h and p_l as the factory price corresponding to the optimistic or "high-value" state, and the fundamental or "low-value" state respectively.

3.1 The before-crash baseline economy

Without credit constraint, agents make decision upon the expected future price. Hence, the before crash system is derived simply by replacing the actual level of future price p_{t+1} with $E_t(p_{t+1})$ in the system ϕ and get the new system, denoted as φ, below.

$$\begin{aligned} \varphi(x_{ht}) &= \begin{bmatrix} \varphi_1(x_{ht}) \\ \varphi_2(x_{ht}) \end{bmatrix} \\ &= \begin{bmatrix} p_{ht+1} \\ f_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{(1+r^*)p_{ht}}{(1-q)(1-\theta)} - \frac{\Omega}{(1-q)(1-\theta)f_t^{\Pi}} - \frac{qp_{lt+1}}{(1-q)} \\ (1-\theta)f_t + \Gamma\left(\left(\frac{1+r^*}{1-\theta}\right)p_{ht} - \frac{\Omega}{(1-\theta)f_t^{\Pi}}\right)^{\Psi} \end{bmatrix} \end{aligned}$$

where $p_{lt+1} = \rho(f_{t+1})$[14].

Given $f_0 \in R_{++}$, the crash time period T, and $p_{h0} > \rho(f_0)$, the before-crash dynamics $\{p_{ht}\}_{t=0}^{T-1}$ and $\{f_t\}_{t=0}^{T-1}$ are determined by the system φ. Remind that φ applies only up to the period $T-1$, just before the crash. During these periods, only the high price p_h is realized ex post.

Next, Proposition 5 characterizes the bubble in regard to the monotonicity of the dynamic of the factory price and stock. This is of the interest because the bubble event is normally 'acknowledged' as the increase in factory price and hence stock over time. Proposition 5 confirms precisely that the present model can replicate this pattern for any given initial price higher than a unique threshold level. Kindly note that the price considered in Proposition 5 is the high-realization price p_h of the system φ, not p from the after-crash system ϕ.

Proposition 5 Under the system φ, for any given (p_{h0}, f_0) where $f_0 \in R_{++}$ and $p_{h0} > \rho(f_0)$, there exists a threshold function $\hat{\rho}(f)$ satisfying the following properties:

1. $\hat{\rho}(f) \geq \rho(f)$ for all $f \in R_{++}$.
2. $\hat{\rho}(f)$ is continuous, strictly decreasing in f over $(0, \bar{f}]$ and strictly increasing in f over $(\bar{f}, \infty)$ where $\bar{f}$ is the valid steady state.
3. For any $t \leq T-1$, $p_{ht+1} > p_{ht}$ and $f_{t+1} > f_t$ if and only if $p_{ht} > \hat{\rho}(f_t)$.
4. For a sufficiently large T, there exists $\hat{t} < T-1$ with $p_{h\hat{t}} > \hat{\rho}(f_{\hat{t}})$, $p_{ht+1} > p_{ht}$, and $f_{t+1} > f_t$ for all $\hat{t} \leq t \leq T-1$. ■

[14] Unlike the after-crash system, the before-crash system φ is stated without the boundary concern due to the reason that the optimistic price, later, turns out to only be higher than or at least equal to the fundamental price which is positive.

Proof See Appendix ■

For $p_{h0} > \hat{\rho}(f_0)$, both factory price and stock keep increasing until the bubble bursts; while, for $p_{h0} > \hat{\rho}(f_0)$, the bubbly price can be temporarily decreasing in the beginning of the bubble path[15]. To sum up, this section shows that the small open economy without credit constraint is a well-constructed platform for the bubble to emerge. Optimistic producers expect the high price for the high capital gain and the expectation is self-fulfilling. With the high price expectation, the contractor builds more factory buildings. However, the bubble is fragile. When the pessimism occurs, the price collapses down to the fundamental level and the economy stays on the fundamental path converging to the steady state[16].

3.2 The before crash economy with the credit constraint

With credit constraint, agents make decision upon the expected future price and the collateral-evaluation price p_{ct+1} which is determined by the financial intermediary. As being assumed to be risk-neutral, the financial intermediary safeguards itself by setting p_{ct+1}, and hence the credit limit, at the level that always guarantees the return of lending equal to the cost of fund, $1 + r^*$. Then, it should rationally evaluate the collateral value at the lowest possible future price level: in general, this implies $p_{ct+1} = \min\{p_{ht+1}, p_{lt+1}\}$ where $p_{lt+1} = \rho(f_{t+1})$[17]. The reason lies on the fact that in the bad realization, the borrower defaults and the financial intermediary obtains the low-valued collateral which is worth less than the actual debt obligation; while, in the good realization the financial intermediary gets no extra benefit since the borrow just repay the debt. Therefore, the ex ante expected return is lower than $1 + r^*$ for any $p_{ct+1} > \min\{p_{ht+1}, p_{lt+1}\}$. This credit limit $(1 - \theta)\min\{p_{ht+1}, p_{lt+1}\}f_t$ is known as a 'natural' limit. The below subsection analyzes the before-crash economy with this credit constraint.

3.2.1 The before-crash natural credit limit

Intuitively, the existence of the bubble induces the increase in the supply of factory buildings. To fuel the bubble, the additional borrowing is required. In the presence of the binding natural collateralized credit limit where $p_{ct+1} = \min\{p_{ht+1}, \rho(f_{t+1})\}$, the increase in the factory supply lowers the fundamental price of factory buildings and hence p_{ct+1}. This lessens the ability of the producer to borrow; as a result, the bubble might be ruled out. Precisely, the

[15] Unlike Weil [25], this decreasing price does not need the dependency between the existence of bubbles and fundamental value.

[16] Note that in Proposition 5 the given $p_{h0} > \rho(f_0)$ is required for the bubble existence of any other period t. In other words, if a rational asset bubble exists, it must have started on the first day of trading. This is consistent with the standard bubble literature (see Diba and Grossman [5, 6], Weil [25], and Jarrow, Protter, and Shimboposits [10])

[17] In the normal bubble context, the lowest possible price level should be the fundamental price since the bubble is usually perceived to be positive. However, technically it might be possible that the endogenous optimistic price p_{ht+1} turns out to be lower than the fundamental price p_{lt+1}.

next proposition states the impossibility of bubble emergence confirming this intuition.

Proposition 6 In the case of Economy 2 with $f_0 \geq \hat{f}$ with the before-crash natural credit limit where $p_{ct+1} = \min\{p_{ht+1}, \rho(f_{t+1})\}$, the bubble cannot exist. ■

Proof The first step is to show that for any $f_t \geq \hat{f}$, the before-crash economy is credit-binding. For the non-binding region, Equation 9 and 10 replace Equation 2 and 4 respectively.

$$f_{t+1} = (1-\theta) f_t + \Gamma E_t(p_{t+1})^{\frac{\alpha}{1-\alpha}} \tag{9}$$

$$\frac{\varepsilon B f_t^{\varepsilon-1} + (1-\theta)\, E_t(p_{t+1})}{p_{ht}} = (1+r^*) \tag{10}$$

where $E_t(p_{t+1}) = (1-q)p_{ht+1} + qp_{lt+1}$ where $p_{lt+1} = \rho(f_{t+1})$.

Therefore, the credit constraint is binding when the following condition holds.

$$\begin{aligned} & (1+r^*)\, b_{2t} = (1+r^*)\,(p_{ht} f_t - W) \geq (1-\theta) p_{ct+1} f_t \\ \Longrightarrow \quad & f_t + \left(\frac{1-\theta}{\varepsilon B}\right) [E_t(p_{t+1}) - p_{ct+1}] f_t \geq \hat{f} \end{aligned}$$

So, if $f_t \geq \hat{f}$, the credit constraint is binding.

When the credit constraint is binding, Equation 11 below (which replaces Equation 6 in the after-crash economy) holds.

$$\frac{\beta(1+r^*)[\varepsilon B f_t^{\varepsilon-1} + (1-\theta)\,(E_t(p_{t+1}) - p_{ct+1}) f_t] + (1-\theta)\, p_{ct+1}}{p_{ht}} = (1+r^*) \tag{11}$$

According to Equation 11, the non-negativity constraint is binding when the following condition holds.

$$\begin{aligned} c_{21t} &= W + \frac{(1-\theta) p_{ct+1} f_t}{(1+r^*)} - p_{ht} f_t \leq 0 \\ \Longrightarrow \quad & f_t + \left(\frac{1-\theta}{\varepsilon B}\right) [E_t(p_{t+1}) - p_{ct+1}] f_t \geq \tilde{f} \end{aligned}$$

Therefore, if $f_t \geq \hat{f}$, both of the credit and non-negativity constraints are binding[18].

The second step is to show that the fundamental price path is a unique solution for the case of Economy 2 with $f_0 \geq \hat{f}$. When both constraints are

[18] Note that $\beta(1+r^*)\varepsilon B f_t^{\varepsilon} > (1+r^*)W$ when the economy is patient means $f_t > \hat{f}$.

binding, Equation 12 (which replaces Equation 5 in the after-crash economy) holds.

$$\frac{(1+r^*)Wf_t^{-1}+(1-\theta)\,p_{ct+1}}{p_{ht}}=(1+r^*) \tag{12}$$

Firstly, consider the system where $p_{ct+1}=p_{lt+1}=\rho(f_{t+1})$. Thereby, the before-crash credit-constrained system (from Equation 9 and 12), denoted as ζ, is the following.

$$\begin{aligned}\zeta(x_{ht}) &= \begin{bmatrix} p_{ht+1} \\ f_{t+1} \end{bmatrix} \\ &= \begin{bmatrix} \zeta_1(x_{ht}) \\ \zeta_2(x_{ht}) \end{bmatrix} = \begin{bmatrix} \frac{1}{(1-q)}\left(\frac{f_{t+1}-(1-\theta)f_t}{\Gamma}\right)^{\frac{1}{\Psi}}-\left(\frac{q}{1-q}\right)\rho(f_{t+1}) \\ \rho^{-1}\left(\left(\frac{1+r^*}{1-\theta}\right)\left(p_{ht}-\frac{W}{f_t}\right)\right) \end{bmatrix}\end{aligned}$$

where $p_{ht}\geq\rho(f_t)=p_{ct}$ for all $t<T$.

As being Economy 2, for any x_{ht} on the globally stable manifold of ϕ_{cb}, the dynamics of ζ is exactly the same as ϕ_{cb}. This implies that the steady state of the above system is $\bar{x}_{cb}$. Moreover, the stable manifold of this system and the fundamental price function $\rho(f_{t+1})$ coincide over the domain $[\hat{f},\infty)$.

Since $f_0>\hat{f}$, given $x'=(\rho(f_0),f_0)$ and any $x''=(\rho(f_0)+\kappa,f_0)$ where $\kappa>0$, $\zeta_1(x')>\zeta_1(x'')$ and $\zeta_2(x')>\zeta_2(x'')$. This means that the dynamics cross the fundamental price path, $p_{ht+1}<\rho(f_{t+1})=p_{ct+1}$, which contradicts $p_{ct+1}=p_{lt+1}=\min\{p_{ht+1},p_{lt+1}\}$.

To see more, linearize ζ at $\bar{x}_{cb}$. The characteristic equation is as follows.

$$|D\zeta(\bar{x}_{cb})-\lambda I|=\lambda^2+[\Xi\Theta+\Delta+\Theta\Phi]\,\lambda-\Xi[(1-\theta)]\Theta=0$$

where $\Xi=\frac{(\theta\bar{f}_{bb})^{\frac{1-\Psi}{\Psi}}}{(1-q)\Psi\Gamma^{\frac{1}{\Psi}}}>0$, $\Theta=-\left(\frac{1+r^*}{1-\theta}\right)\rho^{-1\prime}(\bar{p}_{cb})>0$, $\Delta=\frac{q(1+r^*)}{(1-q)(1-\theta)}>0$, and $\Phi=\frac{W}{f_{bb}^2}>0$[19].

Since $-\Xi[(1-\theta)]\Theta<0$ and, as having been shown, there exists the stable manifold, $0<\lambda_1<1$ and $\lambda_2<0$. This confirms the finding that the dynamics of ζ oscillate across the stable manifold, which is the fundamental price path. Therefore, for any $p_{h0}>\rho(f_0)$, $p_{ct+1}\neq p_{lt+1}$, which is a contradiction.

Alternatively, consider the system where $p_{ct+1}=p_{ht+1}$. It turns out that the system $\phi_{cb}$is obtained. Since ϕ_{cb} is topologically equivalent to ϕ, Proposition 5 states that for any $p_{ht}>\rho(f_t)$, $p_{ht+1}=p_{ct+1}>\rho(f_{t+1})$ for all $t<T$ which contradicts $p_{ct+1}=p_{ht+1}=\min\{p_{ht+1},p_{lt+1}\}$.

As a result, for $p_{ht}>\rho(f_t)$, neither p_{ht+1} nor p_{lt+1} can be p_{ct+1}[20]. This concludes that p_{ht} must be $\rho(f_t)$ which proves that the bubble cannot exist. ■

[19] Note that $\rho^{-1\prime}(\bar{p}_{cb})=\left.\frac{d\rho^{-1}}{dp_{lt+1}}\right|_{p_{lt+1}=\bar{p}_{cb}}$.

[20] Note that for $p_{ht}<\rho(f_t)$, if $p_{ct+1}=p_{ht+1}$, the factory price and stock dynamics will eventually be negative which is not allowed in the forward-looking equilibrium.

The proposition shows that if the financial sector follows this natural credit limit practice, the existence of the credit constraint helps prevent bubbles. Then, the question is what goes wrong in the real world. Throughout the history, many financial crises that the world has experienced indicate that the financial intermediary does not behave in this manner. In the Asian crises, including Japan's bubble burst, the financial intermediary did not lend out according to the lowest possible value of the collateral, but the market value or the expected value instead. This loose lending policy may be justified by banking characteristics in these countries which may give the financial intermediary the extra benefit in the high-realization state; for example, the symbiotic relations between the financial intermediary and industrial corporations, and the equity-linked asset holding of the financial intermediary (see Dubach and Li [7], Radelet, Sachs, Cooper, and Bosworth [19] and Charumilind, Kali, and Wiwattanakantang [4]) In Japan during 1980s, the shift in tendency to rely more on equity and equity-linked financing instead of borrowing from the financial intermediary causes the lending competition more intense and might lead to this risky lending, (see Noguchi [17]).

In what follows, the case which the financial intermediary assesses the collateral at the expected value $p_{ct+1} = E_t(p_{t+1}) > \rho(f_{t+1})$ is assumed. Interestingly, the dramatic difference in outcome is obtained: the credit constraint helps bubbles to emerge.

3.2.2 The before-crash expectedly-valued credit limit

Using $p_{ct+1} = E_t(p_{t+1})$, the before-crash system, written in term of p_h and denoted as $x_{ht+1} = \varphi_j(x_{ht})$ where φ_{jm} for $m = 1, 2$ accordingly below, can be attained simply by replacing p_{t+1} in the after-crash system ϕ_j by $E_t(p_{t+1})$.

$$\begin{aligned} \varphi_j(x_{ht}) &= \begin{bmatrix} \varphi_{j1}(x_{ht}) \\ \varphi_{j2}(x_{ht}) \end{bmatrix} \\ &= \begin{bmatrix} p_{ht+1} \\ f_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{(1+r^*)p_{ht}}{(1-q)(1-\theta)} - \frac{\Omega_{jt}}{(1-q)(1-\theta)f_t^{\Pi}} - \frac{qp_{lt+1}}{(1-q)} \\ (1-\theta)f_t + \Gamma\left(\left(\frac{1+r^*}{1-\theta}\right)p_{ht} - \frac{\Omega_{jt}}{(1-\theta)f_t^{1-\varepsilon}}\right)^{\Psi} \end{bmatrix} \end{aligned}$$

where $p_{lt} = \rho(f_t)$ and $j = non, cb$.

In the presence of the expectedly-valued credit limit, the regime-switching critical value in Proposition 2 is not affected. In other words, here $\hat{f}$ is independent of the crash probability q. So, the before-crash and after-crash systems of the same regime operate on the same region which makes the framework simple to analyze.

Unlike the former section, when the credit limit is set by the expected value of the future collateral in possession, the bubble itself increases the expected price over time and hence enlarges an ability of the producer to borrow. This positive feedback loop between the rising bubble and the ease in borrowing helps fuel the bubble over time. Written analogously to Proposition 5, the next proposition shows that the bubble exists for any $p_{h0} > \rho(f_0)$; moreover, it characterizes the system φ_j to find threshold function $\hat{\rho}(f)$ which results in the increasing dynamics of factory price and stock over before-crash periods. Figure 3a-b show $\hat{\rho}(f)$ in Economy 1 and 2 respectively.

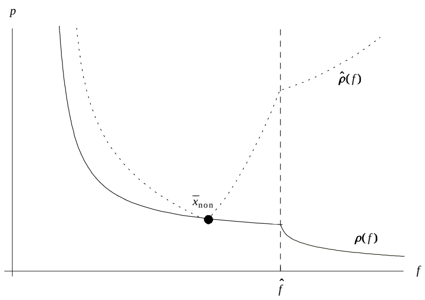

Figure 3a: $\hat{\rho}(f)$ of Economy 1

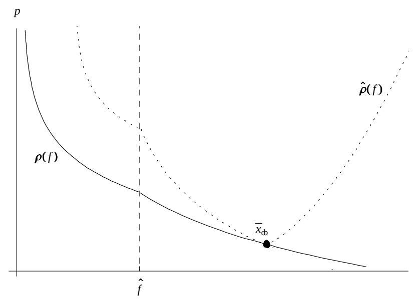

Figure 3b: $\hat{\rho}(f)$ of Economy 2

Proposition 7 Under the regime-dependent system φ_j, for any given (p_{h0}, f_0) where $f_0 \in R_{++}$ and $p_{h0} > \rho(f_0)$, there exists a threshold function $\hat{\rho}(f)$ satisfying the following properties:

1. $\hat{\rho}(f) \geq \rho(f)$ for all $f \in R_{++}$.
2. $\hat{\rho}(f)$ is continuous, strictly decreasing in f over $(0, \bar{f}_j]$ and strictly increasing in f over $(\bar{f}_j, \infty)$ where $\bar{f}_j$ is the valid steady state.
3. For any $t \leq T-1$, $p_{ht+1} > p_{ht}$ and $f_{t+1} > f_t$ if and only if $p_{ht} > \hat{\rho}(f_t)$.
4. For a sufficiently large T, there exists $\hat{t} < T-1$ with $p_{h\hat{t}} > \hat{\rho}(f_{\hat{t}})$, $p_{ht+1} > p_{ht}$, and $f_{t+1} > f_t$ for all $\hat{t} \leq t \leq T-1$. ■

Proof See Appendix. ■

In contrast to the natural credit limit case, the expectedly-valued credit constraint recovers the bubble platform like in the baseline case. Moreover, subject to an exogenous world interest rate shock (the financial liberalization, for example), this credit constraint can help set up the initial bubble $p_{h0} > \rho(f_0)$ naturally.

Endogenising initial price Consider the following scenario. The purchase of factory buildings is proceeded in two lots: the first lot is the old depreciated stock $(1-\theta)\bar{f}$ and the second is the newly-built one $\theta\bar{f}$. Suppose that at period -1, after the producer has purchased the first lot at the price $\bar{p}$, the world interest rate unexpectedly drops from r^* to r^*_n. The producer consequently re-determines his demand for factory holding with regards to the implicit capital gain from purchasing the first lot at the cheaper price. The time line is summarized in Figure 4 below[21].

[21] Note that all the events in the figure happen very closely to the period -1. Only the order of arrows, not the located distance from period -1 and 0, matters.

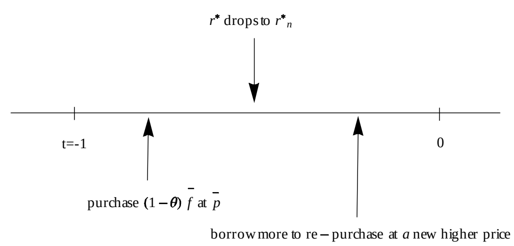

Figure 4: Time line

Since we are dealing with the change in the world interest rate, Proposition 8 below provides the comparative static of various steady states and the critical value with r^*. Note again that in a particular economy only one steady state is valid.

Proposition 8 In general, $\frac{\partial \bar{f}_j}{\partial r^*} < 0$ and $\frac{\partial \hat{f}}{\partial r^*} > 0$ where $j = non, cb$. Furthermore, $\frac{\partial \bar{p}_{non}}{\partial r^*} < 0$ if and only if $\frac{\alpha(1-\varepsilon)}{1-\alpha} < \left(\frac{r^*+\delta}{r^*+\theta}\right)$; while, $\frac{\partial \bar{p}_{cb}}{\partial r^*} < 0$ if and only if $\frac{\alpha}{1-\alpha} < \left(\frac{r^*+\delta}{r^*+\theta}\right)\left(\frac{1-\theta}{1+r^*}\right)$. ■

Proof Trivial. ■

Intuitively, when the decrease in the world interest rate benefits the consumption good production more than the factory construction, it should push up the fundamental price. In particular, an relatively small α compared to ε implies that the capital input relatively produces less output (factory buildings) than the factory input does (consumption good). However, if the world interest rate drops dramatically, the economy may move from non-binding regime to binding regime. Being constrained by the credit limit may prevent the fundamental price to increase as elaborated in Proposition 4. To see this, consider the case of Economy 1 where $\left(\frac{r^*+\delta}{r^*+\theta}\right)\left(\frac{1-\theta}{1+r^*}\right) < \frac{\alpha}{1-\alpha} < \left(\frac{r^*+\delta}{r^*+\theta}\right)\left(\frac{1}{1-\varepsilon}\right)$ and the economy is initially at the steady state $x_{h-1} = \bar{x}_{non}$. Denote subscript n associating variables or functions with the new lower world interest rate r^*_n. When the world interest rate drops, Proposition 8 states that $\bar{p}_{non,n} > \bar{p}_{non}$, $\bar{f}_{j,n} > \bar{f}_j$, and $\hat{f}_j < \hat{f}$ where $j = non, cb$. Suppose that $\bar{p}_{non,n} > \bar{p}_{non}$, $\bar{p}_{cb,n} < \bar{p}_{cb}$, and $\bar{f}_{k,n} > \bar{f}_k$ where $k = non, cb$. Suppose that $f_{-1} > \hat{f}_n$, and $1 - \beta(1 + r^*_n) < 0$. In this case, the economy at period -1 changes from the non-binding regime to the credit-binding regime (Economy 1 to Economy 2). It is possible that even though the $\bar{p}_{non,n} > \bar{p}_{non}$, the new fundamental price $\rho_n(f_{-1})$ is lower than before because the valid fundamental price is no longer the one associated with $\bar{p}_{non,n}$ but the one associated with $\bar{p}_{cb,n}$. This is illustrated in Figure 5 below.

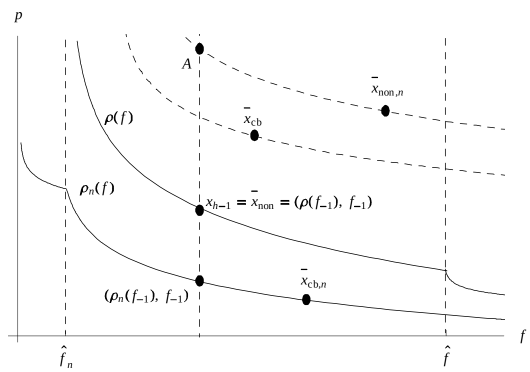

Figure 5: Constrained fundamental price

If this was the baseline case, the new fundamental price would be placed on the point A in the figure. However, the economy is credit constrained and point A cannot be attained. The new fundamental price then depends on the credit-binding system and with $\frac{\alpha(1-\varepsilon)}{1-\alpha} < \left(\frac{r^*+\delta}{r^*+\theta}\right)$ it may be lower than the initial level. In order to generally attain the higher fundamental price as interest rate decreases, $\frac{\alpha}{1-\alpha} < \left(\frac{r^*+\delta}{r^*+\theta}\right)\left(\frac{1-\theta}{1+r^*}\right)$ is required. This implies an unreasonably low α which is empirically inconsistent and it is not assumed here[22].

Hence, the scenario to endogenise the initial bubble is the following. Consider an economy of Economy 1's type where $\frac{\alpha(1-\varepsilon)}{1-\alpha} < \left(\frac{r^*+\delta}{r^*+\theta}\right)$ and is at the steady state $\bar{p}_{non}$ and $\bar{f}_{non}$ initially. Then, the sequence of events as in Figure 4 occurs at period -1. Moreover, assume that the decrease in r^* causes $\hat{f}_n < f_{-1} = \bar{f}_{non}$.

In the micro level, each producer has to re-consider his demand for factory holdings and the new borrowing. Given the sequence of price, $(p_{h-1} - \bar{p}_{non})(1-\theta)\bar{f}_{non}$ is the capital gain and can be considered as an given extra endowment over W. However, this capital gain depends on p_{h-1}. In other words, the level of price the producer expects to occur determines the capital gain and hence the new borrowing-investment decision which in turn determines the level of price in the macro level.

The no-arbitrage condition still holds.

$$\frac{\varepsilon B \acute{f}_{-1}^{\varepsilon-1} + (1-\theta)\, E_{-1}(p_0)}{p_{h-1}} = (1 + r_n^*) \tag{13}$$

As being patient, the producer consumes nothing when he is young. Using $c_{21-1} = 0$, the credit constraint, which takes into account the former debt obligation, is the following..

[22] With the 30-year interest rate $r^* = 17$, $\theta = 0.8$, and $\delta = 1$, it implies that α should be less than 1 percent. Interpreting α as a income share of capital, α is empirically about 30 percent.

$$(1+r^*)\bar{b}_{2-1} + (1+r_n^*)\acute{b}_{2-1} \leq (1-\theta)E_{-1}(p_0)\acute{f}_{-1} \tag{14}$$

where $\bar{b}_{2-1}$ is the existing loan before the crash and $\acute{b}_{2-1}$ is teh new additional loan after ther crash: $\acute{b}_{2-1} = p_{h-1}\acute{f}_{-1} - W - (p_{h-1} - \bar{p}_{non})(1-\theta)\bar{f}_{non} - \bar{b}_{2-1}$.

Then, the question is whether this optimal borrowing $\acute{b}_{2-1}$ violates the credit constraint. Notice that the higher price p_{h-1} the producer takes as given, the higher capital gain he receives and the less credit-constrained he becomes[23].

For the later part of the analysis, firstly conjecture that the producer takes a sufficiently high price p_{h-1}, which makes the credit constraint non-binding, as given. Then, all producers and the financial intermediary plays 'speculative borrowing game' which in turn determines p_{h-1}[24]. For the argument to be consistent, we check that p_{h-1} has to be higher than $\bar{p}_{non}$ for the producer to have the capital gain from the first-lot purchase. At last, numerical test is conducted to show that this p_{h-1} can create p_{h0} which sets up the bubble in the spirit of Proposition 7.

The speculative borrowing game is a two-stage game which the financial intermediary and all producers take part in. At stage-1, the financial intermediary announces the credit limit to the public. At stage-2, the producer decides how much he wants to borrow.

Working backward, the first step is to analyze the decision in stage-2 given the credit limit. If the credit limit is still not reached, there is an incentive for the producer to borrow and invest more: if every producer borrows and invest more, the demand for factory holding with the fixed supply will raise the price p_{h-1} up and increases the capital gain $(p_{h-1} - \bar{p}_{non})(1-\theta)\bar{f}_{non}$ from the first-lot purchase. As long as the no-arbitrage condition (Equation 13) which guarantees the return of factory investment equal to $(1+r_n^*)$ holds, the producer is better off[25].

Definition 4 A strong Nash equilibrium is a Nash equilibrium in which no coalition, taking the actions of its complements as given, can cooperatively deviate in a way that benefits all of its members. ∎

According to the above definition, the only strong Nash equilibrium in the stage-2 game is when every producer borrows and invests at the credit limit since this maximizes the capital gain from the first-lot purchase while Equation 13 still holds. Assuming a strong Nash equilibrium as an equilibrium concept of

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[23] It may seem unclear whether the increase in p_{h-1} increases or decreases $\acute{b}_{2-1}$. However, after substituting $E_{-1}(p_0)$ from Equation 13 in Equation 14, the credit constraint becomes $p_{h-1} \geq \bar{p}_{non} + \frac{(r^*-r_n^*)\bar{b}_{2-1}+\varepsilon B\acute{f}^{\varepsilon}_{-1}-(1+r_n^*)W}{(1+r_n^*)(1-\theta)\acute{f}_{-1}}$. This proves that the higher p_{h-1} is taken as given, the less credit-constrained the producer becomes.

[24] The uniqueness of p_{h-1} depends on the selection of equilibrium definition. The game may have many Nash equilibriums, but the the strong Nash equilibrium is unique.

[25] Note that only symmetric equilibrium can be Nash equilibrium. On one hand, if only one producer borrows and invests more, his action cannot affect the factory price in the macro level and he will end up bearing more cost. On the other hand, if only one producer borrows and invests less than other, his marginal return of factory investment will still be greater than $(1+r_n^*)$ and it is not optimal. As a result, every symmetric borrowing profile which is less than or equal to the credit limit is Nash equilibrium.

this game, the equilibrium of this stage-2 game is where every producer borrows and invests at the credit limit[26].

At stage-1, the financial intermediary expects the outcome from the stage-2 and set the credit limit consistently, assuming the use of the expectedly-valued basis. Remind that the credit limit $(1-\theta)E_{-1}(p_0)\acute{f}_{-1}$ is endogenous: it relies on $E_{-1}(p_0)$ which relates to p_{h-1} via Equation 13. So, p_{h-1} can be determined below by the credit constraint using the results that the producer borrows at the credit limit, the financial intermediary sets the limit consistently, and the factory market clears $\acute{f}_{-1} = \bar{f}_{non}$.

$$p_{h-1} = \bar{p}_{non} + \frac{(r^* - r_n^*)\bar{b}_{2-1} + \varepsilon B \bar{f}_{non}^{\varepsilon} - (1+r_n^*)W}{(1+r_n^*)(1-\theta)\bar{f}_{non}} \tag{15}$$

Since $\hat{f}_n < f_{-1} = \bar{f}_{non}$ in this scenario, this implies $\varepsilon B \bar{f}_{non}^{\varepsilon} > (1+r_n^*)W$. Consequently, $p_{h-1} > \bar{p}_{non}$ which means the producer really receives the capital gain as conjectured. Note that the dynamics from period -1 to period 0 follow $\varphi_{non,n}$ since, in our argument, Equation 13 is valid. Then, $(p_{h0}, f_0) = \varphi_{non,n}(p_{h-1}, f_{-1})$. If $p_{h0} > \rho(f_0)$, the bubble is successfully set up according to Proposition 7.

Before moving to the numerical experiment to test whether $p_{h0} > \rho(f_0)$, let us summarize the argument so far. For arbitrary given price, the producer derives his optimal borrowing. If this borrowing is still less than the credit limit, he has an incentive to borrow and invest more along with every other producer and this will change the price he takes as given. Simultaneously, we end up with the price p_{h-1} which maximizes the capital gain while being given for the expected utility maximization. This price is higher than the price of the first-lot purchase. This is consistent with the argument that the producer expects capital gain and invests more even though the fundamental price may drop as discussed earlier in Figure 5.

Lastly, the numerical test is conducted to prove the potential of this argument to endogenise the initial bubble[27]. The result of the experiment is that,

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[26] Borrowing up to the limit is a natural solution in this credit constraint context. Without credit limit, this argument is not well-defined since the maximized strategy would be that everyone demands infinite borrowing. This feature of credit constraint is as highlighted by Kochelakota [13] who argues that the loan upper limit is needed for the infinite wealth problem.

[27] Since the framework in this paper is the overlapping generations model with two-period-lived agent, this translates one period's length into 30 years. Given $\frac{\alpha(1-\varepsilon)}{1-\alpha} < \left(\frac{r^*+\delta}{r^*+\theta}\right)$ and $\beta(1+r^*) > 1$, let $\beta = 0.3$, $r^* = 17$, $\delta = 1$, $\theta = 0.8$, $A = B = 1$, and $\alpha = \varepsilon = 0.3$ which implies that the yearly rates are common as in the literature: $\beta \approx 0.96$, $r^* \approx 0.1$, $\delta \approx 0.3$, and $\theta \approx 0.05$. Moreover, $r_n^* \backsim unfrnd(3, 17)$, $\frac{N_1}{N_2} \backsim unfrnd(0, 2)$, and $W \backsim unfrnd(0, \bar{p}_{non}\bar{f}_{non})$ where $z \backsim unfrnd(x, y)$ means z is randomly picked from the uniform distribution over (x, y). Recall $\bar{b}_{2-1}$ is the borrowing before the world interest rate shock which provides a degree of freedom to choose. As in Equation 15, the higher $\bar{b}_{2-1}$ is, the higher p_{h-1} becomes. So, the highest reasonable level of $\bar{b}_{2-1}$ is used in the experiment. Choose $\bar{b}_{2-1} = \min\{(1-\theta)\bar{p}_{non}\bar{f}_{non}, \bar{p}_{non}\bar{f}_{non} - W\}$. This means that the producer borrows all up to purchase the first lot if it does not exceed the entire planned borrowing $\bar{p}_{non}\bar{f}_{non} - W$.

The algorithm is as follows. the experiment is conducted for 100 times. In each time, 100,000 samples are drawn. Then, each sample is filtered whether it matches the scenario argued on average, 79 percent of all admissible cases has $p_{h0} > \rho(f_0)$ which implies the great likelihood of our argument to set up the bubble. Note that the argument does not require any asymmetric information as a friction to create the bubble. The credit constraint naturally induces the speculative borrowing game and pinpoint p_{h-1}. Insightfully, the credit constraint constrains the fundamental price to the low level which makes it easier for the bubble to emerge (easier for $p_{h0} > \rho_n(f_0)$). To sum up, Proposition 9 is addressed below.

Proposition 9 In the presence of the expectedly-valued credit constraint, the unexpected capital gain and the speculative borrowing game can put the credit-constrained economy on the unique non-stationary sunspot path established in Proposition 7. ■

Proof As argued and shown numerically in the text. ■

4 Boom, crash, overutilization and prolonged recession

Consider the economy without the credit constraint or with the expectedly-valued credit constraint given $p_{h0} > \rho(f_0)$. The interesting case is when T is large enough to push the factory accumulation further away from the steady state level before the crash. Figure 6 below illustrates the phenomenon in the case of expectedly-valued credit constrained economy.

---

earlier: $\hat{f}_n < f_{-1} = \bar{f}_{non} < \hat{f}$. Considering only samples that replicate the scenario, calculate p_{h-1} according to Equation 15. After that, compute $(E_{-1}(p_0), f_0) = \hat{\phi}_{non,n}(p_{h-1}, f_{-1})$. Next, iterate $\hat{\phi}_{cb,n}^{\{m\}}(E_{-1}(p_0), f_0)$ for $m = 1, 2, ..., 1000$. If this gives an eventually increasing sequence of factory price, this implies that actually $p_{h0} > \rho(f_0)$ and the economy is set on the bubble path. Lastly, compute the probability of the samples where $p_{h0} > \rho(f_0)$ conditional on the samples where $\hat{f}_n < f_{-1} = \bar{f}_{non} < \hat{f}$ and average over 100 experiments.

Note that in the algorithm $\hat{\phi}_{non,n}$ is used instead of $\varphi_{non,n}$. The difficulty of using $\varphi_{non,n}$ lies on finding the fundamental price $\rho(f)$ numerically. However, from the analytical result, $E_t(p_{t+1}) > \rho(f_{t+1})$ if and only if $p_{ht+1} > \rho(f_{t+1})$. If $\hat{\phi}_{cb,n}^{\{m\}}(E_{-1}(p_0), f_0)$ results in the eventually exploding price path, this must imply $\varphi_{cb,n}^{\{m\}}(p_{h0}, f_0)$ have the same qualitative path as well. Hence, $p_{h0} > \rho(f_0)$.

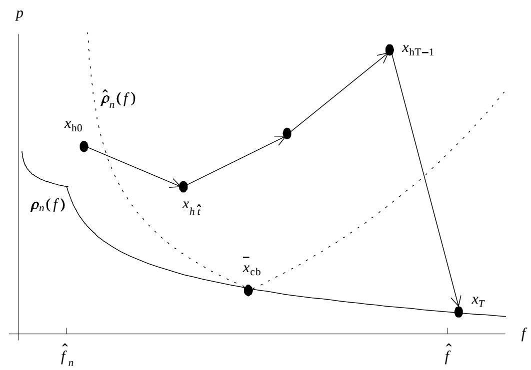

Figure 6: Boom-bust episode

In such case, the real GDP eventually keeps soaring before the crash causing the boom. To see this, the real GDP or $RGDP_t$ for the before-crash economy is given below[28].

$$RGDP_t = \Gamma\left[E_{t-1}(p_t)\right]^{\Psi} + \varepsilon B f_{t-1}^{\varepsilon}$$

According to Proposition 6, f_{t-1} is increasing before the crash; while, although $E_{t-1}(p_t)$ may temporarily fall for some early periods, it will finally continue rising and so will the real GDP. At time T, the sharp fall in price causes the widespread default and brings about the great loss to the financial intermediary. The after-crash real GDP formula is slightly changed as follows.

$$RGDP_t = \Gamma p_t^{\Psi} + \varepsilon B f_{t-1}^{\varepsilon}$$

Actually, the period-T output from both sectors still increases from the last period since the capital and factory stock is predetermined. However, the significant fall in price will greatly affect the production of both factory buildings and consumption good in period $T+1$ due to the lower profitability and the reduced ability to borrow.

Before the crash, the optimistic belief in the high but risky capital gain drives the economy with the miraculous growth rate. The massive factory construction and significant capital gain result. After the crash, the economy encounters a very deep recession. The over-construction of factory buildings relative to the steady state level during the before-crash period leads to the factory overutilization and hence the very low fundamental price appears[29]. The factory production has to be reduced tremendously. Subsequently, the factory stock will be left to be depreciated over time until the total stock is reduced to the steady

[28] Here real GDP is the sum of factory buildings and the consumption good produced in the period t, divided by the weighted-average price index, which is the simple weighted-average price index $\frac{\sum_{g\in G} p_{gt}Q_{gt}}{\sum_{g\in G} Q_{gt}}$ where G is a set of all goods and Q_{gt} is the quantity of good g produced in time t.

[29] Glaeser Gyourko and Saiz [9] also have the similar result but only in the irrational bubble setting.

state level. These factory overutilization and prolonged recession result from the sunspot boom session that pushes the economy too far away from where it should have been.

Note that after the crash the price is increasing while the factory stock is decreasing to the steady state. Indeed, there are two contradicting effects to the real GDP. The first effect is the price effect which values the factory more[30]. The second one is the stock effect which cuts down consumption good production. Nevertheless, the economy will eventually reach the steady state where the real GDP will be lower than the level of the boom period. Figure 7 shows the time path of real GDP corresponding to the economy in Figure 6 . The two contradicting effects result in the shaded area where the trend of the path of real GDP after the crash is ambiguous.

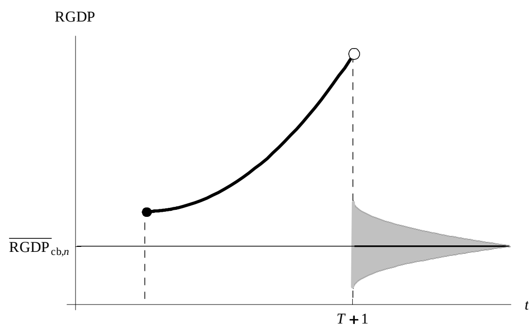

Figure 7: Real GDP

These patterns from Figure 6 and 7 are empirically consistent with many financial crises. According to Dubach and Li [7], the real estate price in Japan was phenomenally skyrocketed during 1980s and then crashed in 1991. In the boom period, Japan experienced the continuous high growth rate; but then suffered from the low L-shaped growth pattern for a decade afterward. For other emerging markets, Thailand's and Indonesia's real estate price seemed to be stable during early 1990s but still encountered the sudden crash in 1997. From the present model, this implies that the crash happened quite early relatively to Japan's case: $T \leq \hat{t}$ in Figure 6. Despite the stable real estate price, the increase in the supply helps expanded the growth rate of these countries rapidly (see Radelet, Sachs, Cooper, and Bosworth [19]).

5 Policy implication

It is shown that the small open economy relaxes the wealth constraint of agents by having a large amount of foreign capital inflow at the low cost, which makes the economy fragile to the bubble episode. Apart from limiting the capital

[30] Note that $RGDP_t$ is a function of price since the contractor invests the capital regarding the next period price of factory buildings. Thus, the price effect is that when price is expected to be high, the contractor invests more capital and produces more new factory next period.

account liberalization which directly eliminates the ground for the bubble to emerge, this paper strikingly finds the double-bladed role of credit constraint which can either solve or worsen the bubble problem. Proposition 6 suggests that the proper credit constraint can help rule out the bubble[31]. However, Proposition 7 and the corresponding initial-bubble-endogenising scheme imply that if the credit is set higher than the fundamentally-valued level, the outcome totally becomes in favor of the bubble emergence, even more severe than the no-credit-constraint case. Therefore, if the bubble is decided to be undesirable, the policy implication is straightforwardly that the policymaker should focus on tightening the regulation and monitoring the financial sector. Supervision on the debt contract, especially on the collateral requirement, is really crucial. If all financial intermediaries strictly set the credit limit pessimistically (at most equal to the natural credit limit level $(1+r^*)b_{2t} \leq (1-\theta)\rho(f_{t+1})f_t$), this automatically re-creates the wealth constraint on agents to finance bubbles in the future and rules out bubbles in the forward looking manner.

Nevertheless, any abovementioned policy has its cost. That is, it lowers the fundamental level of real GDP compared to the baseline model: Proposition 4 shows that the presence of collateralized credit constraint constrains the fundamental price path to the lower level and hence lower real GDP

6 Conclusion

The small open economy environment supports the emergence of a bubble by allowing the large amount of capital inflow to fuel the bubble as optimism goes on until the crash. With the collateralized credit constraint, the situation can be substantially different. If the natural credit limit is applied, this prevents the bubble to emerge. In particular, the bubble induces the over-construction of factory buildings which lowers the fundamental price and hence the credit limit. This eliminates an ability to fuel the bubble in the future and the bubble then cannot emerge today.

However, if the expectedly-valued credit limit is applied, the bubble can still exist while the economy has to bear more cost as the fundamental becomes constrained. Moreover, this credit constraint can help endogenise the initial bubble, via the "speculative borrowing game", subject to the unexpected drop in the world interest rate, for example the financial liberalization preceding most financial crises in many emerging countries. The bubble causes the boom-bust episode as it grows and crashes. The crash is sudden and sharp as the over-construction and overutilization of factory buildings depress the fundamental value to the very low level. The succeeding recession results from the reduction in the factory stock in the process of adjusting the economy back to the steady state.

For policy implication, imposing the natural credit constraint can prevent or terminate the bubble. Yet, the double-bladed role of the credit constraint

[31] Note that interpreting emerging countries as having low initial endowment (W is small), these countries most likely fall into Proposition 6's context.

reminds a policy maker to be careful in implementing the policy. Tightening the regulation and monitoring the financial sector on the credit limit are remarkably crucial.

7 Appendix

Proof of Proposition 1 Consider the system ϕ without boundary, $\hat{\phi} : \Theta \to R^2_{++}$ where $\Theta \in R^2_{++}$ is a neighborhood of $\bar{x}$ and $\hat{\phi}_m : \Theta \to R_{++}$ for $m = 1, 2$ is defined below accordingly.

$$\hat{\phi}(x_t) = \begin{bmatrix} \hat{\phi}_1(x_t) \\ \hat{\phi}_2(x_t) \end{bmatrix} = \begin{bmatrix} p_{t+1} \\ f_{t+1} \end{bmatrix} = \begin{bmatrix} \left(\frac{1+r^*}{1-\theta}\right) p_t - \frac{\Omega}{(1-\theta)f_t^{\Pi}} \\ (1-\theta)f_t + \Gamma p_{t+1}^{\Psi} \end{bmatrix}$$

Linearize the system $\hat{\phi}$ at the positive steady state $\bar{x}$. The characteristic equation is as follows.

$$\left| D\hat{\phi}(\bar{x}) - \lambda I \right| = \lambda^2 - \left[\left(\frac{1+r^*}{1-\theta}\right) + (1-\theta) + \frac{\Pi\Omega\Psi\Gamma\bar{p}^{\Psi-1}}{(1-\theta)\bar{f}^{\Pi+1}} \right] \lambda + (1+r^*) = 0$$

Define

$$0 < \varphi = \frac{1}{2}\left[\sqrt{\acute{\Delta}^2 - 4(1+r^*)} - \grave{\Delta} \right] < 1$$

where

$$\acute{\Delta} = \left(\frac{1+r^*}{1-\theta}\right) + (1-\theta) + \frac{\Pi\Omega\Psi\Gamma\bar{p}^{\Psi-1}}{(1-\theta)\bar{f}^{\Pi+1}}$$

$$\grave{\Delta} = \left(\frac{1+r^*}{1-\theta}\right) - (1-\theta) + \frac{\Pi\Omega\Psi\Gamma\bar{p}^{\Psi-1}}{(1-\theta)\bar{f}^{\Pi+1}}$$

Eigenvalues λ and corresponding eigenvectors v of $D\hat{\phi}(\bar{x})$ are the following.

$$\lambda_1 = (1-\theta) - \varphi,\ v_1 = \begin{bmatrix} \frac{\Pi\Omega\Psi\Gamma\bar{p}^{\Psi-1}}{(1-\theta)\bar{f}^{\Pi+1}} + \varphi \\ -\left(\frac{1+r^*}{1-\theta}\right)\Psi\Gamma\bar{p}^{\Psi-1} \end{bmatrix}$$

$$\lambda_2 = \left(\frac{1+r^*}{1-\theta}\right) + \frac{\Pi\Omega\Psi\Gamma\bar{p}^{\Psi-1}}{(1-\theta)\bar{f}^{\Pi+1}} + \varphi,\ v_2 = \begin{bmatrix} \frac{\Pi\Omega}{(1-\theta)\bar{f}^{\Pi+1}} \\ \frac{\Pi\Omega\Psi\Gamma\bar{p}^{\Psi-1}}{(1-\theta)\bar{f}^{\Pi+1}} + \varphi \end{bmatrix}$$

Let E^s and E^u denote, respectively, the stable and unstable eigenspaces of the associated Jacobian matrix. Since $0 < \lambda_1 < 1$ and $\lambda_2 > 1$, $E^s = v_1$ and $E^s = v_2$. According to the Centre Manifold Theorem (see Lines [15]), the mapping $\hat{\phi}$ defined on the positive domain is trivially smooth and invertible, so it is diffeomorphism. Hence, there exists a locally stable manifold $W^s_{loc}(\bar{x})$ and a locally unstable manifolds $W^u_{loc}(\bar{x})$ around the steady state $\bar{x}$.

$$W^s_{loc}(\bar{x}) = \left\{ x \in \eta | \lim_{n\to\infty} d[\hat{\phi}^{\{n\}}(x), \bar{x}] = 0 \text{ and } \hat{\phi}^{\{n\}}(x) \in \eta\ \forall n \geq 0 \right\}$$

$$W^u_{loc}(\bar{x}) = \left\{ x \in \eta | \lim_{n\to\infty} d[\hat{\phi}^{-1\{n\}}(x), \bar{x}] = 0 \text{ and } \hat{\phi}^{-1\{n\}}(x) \in \eta \ \forall n \geq 0 \right\}$$

where

$$\hat{\phi}^{-1}(x_{t+1}) = \begin{bmatrix} \hat{\phi}_1^{-1}(x_{t+1}) \\ \hat{\phi}_2^{-1}(x_{t+1}) \end{bmatrix} = \begin{bmatrix} p_t \\ f_t \end{bmatrix} = \begin{bmatrix} \left(\frac{1-\theta}{1+r^*}\right) p_{t+1} + \frac{\Omega(1-\theta)^{\Pi}}{(1+r^*)\left(f_{t+1}-\Gamma p^{\Psi}_{t+1}\right)^{\Pi}} \\ \frac{f_{t+1}}{(1-\theta)} - \frac{\Gamma p^{\Psi}_{t+1}}{(1-\theta)} \end{bmatrix}$$

According to Galor [8], the global stable manifold of $\hat{\phi}$ can be obtained below.

$$W^s(\bar{x}) = \cup_{n\in N} \left\{ \hat{\phi}^{-1\{n\}}(W^s_{loc}(\bar{x})) \right\}$$

$$W^u(\bar{x}) = \cup_{n\in N} \left\{ \hat{\phi}^{\{n\}}(W^u_{loc}(\bar{x})) \right\}$$

Note that since $W^k_{loc}(\bar{x})$ where $k = s, u$ is connected; hence, so is $W^k(\bar{x})$.

The Centre Manifold Theorem states that $W^s_{loc}(\bar{x})$ forms the curve tangent to E^s. Since E^s is negative sloping in the plane $f - p$, $W^s_{loc}(\bar{x})$ is the curve in the neighborhood of $\bar{x}$ where p is decreasing in f. Moreover, for any two points $x_1, x_2 \in W^s(\bar{x})$ where $p_1 > p_2$ and $f_1 < f_2$, $\hat{\phi}_1^{-1}(x_1) > \hat{\phi}_1^{-1}(x_2)$ and $\hat{\phi}_2^{-1}(x_1) < \hat{\phi}_2^{-1}(x_2)$. Therefore, $W^s(\bar{x})$ is strictly decreasing in f. In the similar manner, $W^u(\bar{x})$ is strictly increasing in f[32].

As a result, $W^s(\bar{x})$ generates a fundamental price function $\rho : R_{++} \to R_{++}$ which is unique, continuous, and strictly decreasing in f. ■

Proof of Proposition 3 Consider ϕ_j without the boundary, $\hat{\phi}_j : \Theta_j \to R^2_{++}$ where $\Theta_j \in R^2_{++}$ is a neighborhood of $\bar{x}_j$ and $\hat{\phi}_{jm} : \Theta_j \to R_{++}$ for $m = 1, 2$ as defined accordingly below.

$$\hat{\phi}_j(x_t) = \begin{bmatrix} \hat{\phi}_{j1}(x_t) \\ \hat{\phi}_{j2}(x_t) \end{bmatrix} = \begin{bmatrix} p_{t+1} \\ f_{t+1} \end{bmatrix} = \begin{bmatrix} \left(\frac{1+r^*}{1-\theta}\right) p_t - \frac{\Omega_{jt}}{(1-\theta)f_t^{\Pi}} \\ (1-\theta)f_t + \Gamma p^{\Psi}_{t+1} \end{bmatrix}$$

$$\hat{\phi}_j^{-1}(x_{t+1}) = \begin{bmatrix} \hat{\phi}_1^{-1}(x_{t+1}) \\ \hat{\phi}_2^{-1}(x_{t+1}) \end{bmatrix} = \begin{bmatrix} p_t \\ f_t \end{bmatrix} = \begin{bmatrix} \left(\frac{1-\theta}{1+r^*}\right) p_{t+1} + \frac{\Omega_{jt}(1-\theta)^{\Pi}}{(1+r^*)(f_{t+1}-\Gamma p^{\Psi}_{t+1})^{\Pi}} \\ \frac{f_{t+1}}{(1-\theta)} - \frac{\Gamma p^{\Psi}_{t+1}}{(1-\theta)} \end{bmatrix}$$

Several claims are required for the main proof. Note that the first three claims are direct consequences of the fact that $\hat{\phi}_j$ for $j = non, cb$ is qualitatively equivalent to $\hat{\phi}$

Claim 1 There exist a stable manifold W^s_j and an unstable manifold W^u_j of $\hat{\phi}_j$ forming a strictly decreasing and strictly increasing price function of f over R_{++} respectively. ■

[32] $W^u(\bar{x})$ will be used in the proof of Proposition 3 below.

Proof Since $\hat{\phi}_j$ for $j = non, cb$ is qualitatively equivalent to $\hat{\phi}$. The claim is already proved in the proof of Proposition 1. ■

Claim 2 The dynamics of $\hat{\phi}_j$ do not cross W_j^s and W_j^u. ■

Proof Consider Figure 8 below.

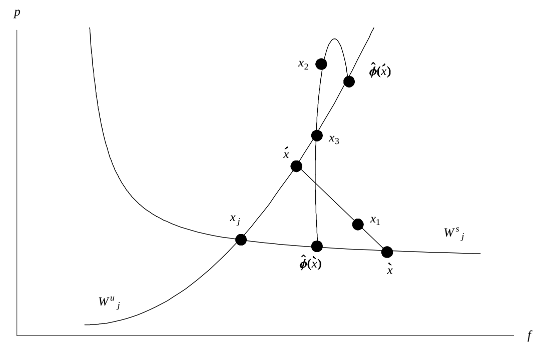

Figure 8: Derivation

Suppose that the dynamics of $\hat{\phi}_j$ move a point x_1, which is not on any manifold, to a point x_2 across the manifold W_j^u as in Figure 8. Pick two arbitrary point $\grave{x}$ and $\acute{x}$ on a part of W_j^s and W_j^u, respectively, which bound the area where x_1 is located. Draw an arbitrary line connected point $\grave{x}$ and $\acute{x}$ in such a way that x_1 is on the interior of the line and this line do not cross any manifold except the end points. Denote this line as arc $\grave{x}\acute{x}$. Then, x_2 must be on $\hat{\phi}(\grave{x}\acute{x})$. Since $\hat{\phi}_j$ is continuous and $\grave{x}\acute{x}$ is a connected set, $\hat{\phi}(\grave{x}\acute{x})$ is connected. Consequently, as in Figure 8, there must exists at least an intersection x_3 between the arc $\hat{\phi}(\grave{x}\acute{x})$ and a manifold. However, x_3 comes from the interior of $\grave{x}\acute{x}$. The existence of x_3 contradicts the definition of the manifold whose dynamics cannot be off the manifold forward and backward. ■

Definition 5 For a given dynamic system $(a_{t+1}, b_{t+1}) = \omega(a_t, b_t)$,

- (a) $\Delta a_{\omega t+1} = a_{t+1} - a_t$ and $\Delta b_{\omega t+1} = b_{t+1} - b_t$ are defined as functions of (a_t, b_t).
- (b) $\Delta a_{\omega t} = a_t - a_{t-1}$ and $\Delta b_{\omega t} = b_t - b_{t-1}$ are defined as functions of (a_t, b_t). ■

Note that the $\Delta a_{\omega t}$ is an inverse transformation of $\Delta a_{\omega t+1}$, not a lag.

Claim 3 The dynamics of $\hat{\phi}_j$ qualitatively follow the phase diagram in Figure 9 below[33]. ■

[33] The dotted single-headed arrow informs the succeeding direction of a point in a particular area while the dashed double-headed arrow inform the preceding direction of that same point.

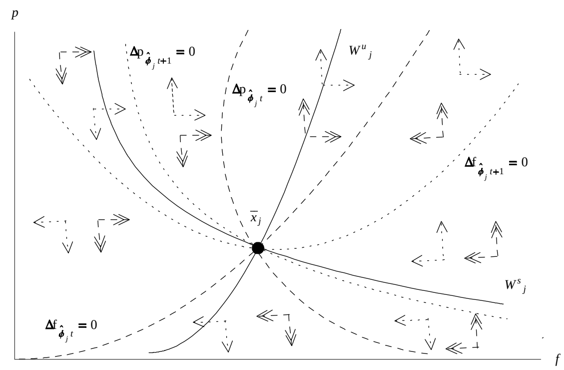

Figure 9: Full phase diagram of $\hat{\phi}_j$

Proof According to Definition 5, the following loci and associated directional field are obtained.

$$\Delta p_{\hat{\phi}_j t+1} \lesseqqgtr 0 \text{ if } \left(\frac{r^* + \theta}{1-\theta}\right) p_t \lesseqqgtr \frac{\Omega_j}{(1-\theta) f_t^{\Pi}}$$

$$\Delta f_{\hat{\phi}_j t+1} \lesseqqgtr 0 \text{ if } \Gamma[\left(\frac{1 + r^*}{1-\theta}\right) p_t - \frac{\Omega_j}{(1-\theta) f_t^{\Pi}}]^{\Psi} \lesseqqgtr \theta f_t$$

$$\Delta p_{\hat{\phi}_j t} \lesseqqgtr 0 \text{ if } \left(\frac{r^* + \theta}{1+r^*}\right) p_t \lesseqqgtr \frac{\Omega_j}{(1+r^*)} \left(\frac{1-\theta}{f_t - \Gamma p_t^{\Psi}}\right)^{\Pi}$$

$$\Delta f_{\hat{\phi}_j t} \lesseqqgtr 0 \text{ if } \frac{\Gamma p_t^{\Psi}}{1-\theta} \lesseqqgtr \frac{\theta f_t}{1-\theta}$$

Figure 9 results trivially. ■

Claim 4 For any $x_t = (p_t, f_t)$ where $f_t = \hat{f}$, $\hat{\phi}_{non}(x_t) = \hat{\phi}_{cb}(x_t)$. ■

Proof By substituting $\hat{f}$ into $\hat{\phi}_j$ accordingly, the claim results. ■

Claim 4 states that at the critical value which separates two regimes, the two systems of each regime coincide.

The main proof Only the case of Economy 1 is proved here. The proof in the case of Economy 2 can be done in the similar manner.

By definition of the fundamental price function, the stable manifold W^s_{non} clearly forms the fundamental price function $\rho(f)$ over $(0, \hat{f}]$. The task is to extend the function over $(\hat{f}, \infty)$ by iterating a selected part of W^s_{non} backward using $\hat{\phi}_{cb}^{-1}$.

For this particular case, let $\{x'\} = W^s_{non} \cap W^u_{cb}$ and $\{x''\} = W^s_{non} \cap \{x = (p, f) \in R^2_{++} | f = \hat{f}\}$. Note that these sets are singleton due to Claim 1. Denote $x''' = \hat{\phi}_{non}(x'')$. According to Claim 4, $x''' = \hat{\phi}_{cb}(x'')$ as well. Moreover, let $\{x''''\} = \{x = (p, f) \in R^2_{++} | \Delta p_{\hat{\phi}_{cb} t+1} = 0 \textit{ and } f = \hat{f}\}$ which is also singleton due to Claim 3. The following claims hold.

Claim 5 In the case of Economy 1 with above notation, for $x' = (p', f')$ and $x''' = (p''', f''')$, $p' > p'''$ and $f' < f'''$. ■

Proof Since $\bar{f}_{non} < \bar{f}_{cb} < \hat{f}$, $f' \in (0, \hat{f})$ by Claim 1. Since $x''' = \hat{\phi}_{cb}(x'')$, the jump from x'' to x''' cannot cross W^u_{cb} due to Claim 2. Thus, this implies $p' > p'''$ and $f' < f'''$, see Figure 10a. ■

Claim 6 In the case of Economy 1, W^s_{non} and W^s_{cb} never cross each other over $(0, \hat{f}]$. ■

Proof Suppose there exists a common point $\breve{x}$ between W^s_{non} and W^s_{cb} over $(0, \hat{f})$. Since $\Omega_{non,t} < \Omega_{cb,t}$, $\hat{\phi}_{non1}(\breve{x}) > \hat{\phi}_{cb1}(\breve{x})$ and $\hat{\phi}_{non2}(\breve{x}) > \hat{\phi}_{cb2}(\breve{x})$. This implies that iterating $\breve{x}$ through $\hat{\phi}_{cb}$ brings the dynamics down further below W^s_{non}. As a result, $\lim_{n\to\infty} \hat{\phi}^{\{n\}}_{cb}(\breve{x}) \neq \bar{x}_{cb}$ whose $\bar{p}_{cb} > \bar{p}_{non}$ and $\bar{f}_{cb} > \bar{f}_{non}$ which results in a contradiction.

Suppose W^s_{non} and W^s_{cb} cross at $\hat{f}$. Then the next forward iteration gives the other common point over $(0, \hat{f})$ which results in a contradiction. ■

Claim 7 In the case of Economy 1 with above notation, for $x'' = (p'', f'')$ and $x'''' = (p'''', f'''')$, $p'''' < p''$. ■

Proof According to Claim 4, the $\Delta p_{\hat{\phi}_{non}t+1} = 0$ locus and the $\Delta p_{\hat{\phi}_{cb}t+1} = 0$ locus coincide at $\hat{f}$. Thereby, from Claim 3, the claim results. ■

Define A_0B_0 as a connected arc of W^s_{non} between point A_0 and B_0 including end points, see Figure 3a.

Next, proceed the following algorithm.

1. Select A_0B_0 by choosing $A_0 = x'''$ and $B_0 = x''$.
2. Iterate $A_{n-1}B_{n-1}$ via $\hat{\phi}^{-1}_{cb}$ to get A_nB_n for $n \in N$ which is defined below.

$$A_nB_n = \hat{\phi}^{-1}_{cb}(A_{n-1}B_{n-1}) = \{x \in R^2_{++} | x = \hat{\phi}^{-1}_{cb}(y) \text{ where } y \in A_{n-1}B_{n-1}\}$$

Claim 8 below characterizes the topological feature of the iteration.

Claim 8 In the case of Economy 1 with above notation, for any $(p, f) \in A_nB_n$ for $n \in N$, $\breve{p} > p$ where $(\breve{p}, f) \in W^s_{non}$. ■

Proof Since $f_t > \hat{f}$ implies $\Omega_{non,t} > \Omega_{cb,t}$, for any $(p, f) \in \hat{\phi}^{-1}_{cb}(A_0B_0)$, $\breve{p} > p$ where $(\breve{p}, f) \in W^s_{non}$. Trivially, $\hat{\phi}^{-1\{n\}}_{cb}(A_0B_0)$ is also located down below W^s_{non} for $n \in N$. ■

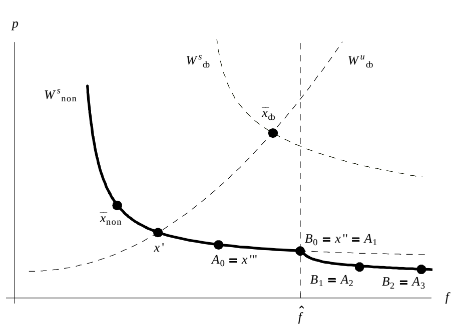

Figure 10a: Derivation

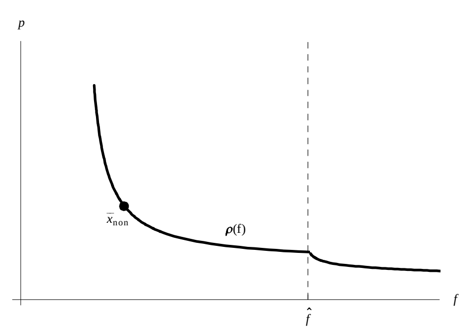

Figure 10b: $\rho(f_t)$ of Economy 1

Figure 11a illustrates the algorithm above[34]. Claim 5 and 6 imply that A_0B_0 lies between W^s_{cb} and W^u_{cb} which implies that the backward dynamics of each arc A_nB_n for $n \in N$ go toward the cb regime. Since $\frac{\partial \hat{\phi}^{-1}_{cb1}}{\partial p_{t+1}} > 0$, $\frac{\partial \hat{\phi}^{-1}_{cb1}}{\partial f_{t+1}} < 0$, $\frac{\partial \hat{\phi}^{-1}_{cb2}}{\partial p_{t+1}} < 0$, $\frac{\partial \hat{\phi}^{-1}_{cb2}}{\partial f_{t+1}} > 0$, and A_0B_0 forms a decreasing price function of f over the corresponding connected support, each of the arc A_nB_n for $n \in N$ forms a decreasing price function of f over each corresponding connected support. Since $B_0 = \hat{\phi}^{-1}_{cb}(A_0)$, $A_n = B_{n-1}$ for $n \in N$. Denote $\rho(f)$ as a fundamental price function formed by W^s_{non} over $(0, \hat{f}]$ together with $\cup_{n \in N} A_nB_n$: $\rho(f)$ is unique, continuous, and strictly decreasing in f.

Lastly, to show that $\rho(f)$ is defined over R_{++}, note that the dynamics of $\hat{\phi}^{-1}_{cb}$ follow the inverse directional field of the phase diagram in Claim 3. Then, Claim 2 and 3 show that for any x lies between W^u_{cb} and the $\Delta p_{\hat{\phi}_{bb}t+1} = 0$ locus, the backward dynamics move in the south-east direction and are always contained between these two loci. From Claim 6 and 7, point A_0 is above the $\Delta p_{\hat{\phi}_{bb}t+1} = 0$ locus but below W^s_{cb}. Hence, the entire arc A_1B_1 lies between W^u_{cb} and the $\Delta p_{\hat{\phi}_{cb}t+1} = 0$ locus and so does $A_{n+1}B_{n+1}$ for $n \in N$. Since the $\Delta p_{\hat{\phi}_{cb}t+1} = 0$ locus does not cross below the $p = 0$ axis, $\rho(f)$ covers R_{++}, see Figure 10b. ■

Proof of Proposition 5 and 7 Since $\hat{\phi}$ is $\hat{\phi}_{non}$, the proof of Proposition 5 is embedded in the proof of Proposition 7 in the case of Economy 1: just consider the whole space to be non-binding region. So, only the proof of Proposition 7 in the case of Economy 1 is presented below. The case of Economy 2 can be proved in the similar manner.

For this particular case, denote $\acute{\rho}(f_t)$, $\grave{\rho}(f_t)$, and $\breve{\rho}(f_t)$ from the $\Delta p_{h\varphi_{non}t+1} = 0$,.$\Delta f_{\varphi_{non}t+1} = 0$, and $\Delta f_{\varphi_{cb}t+1} = 0$ loci respectively.

$$\Delta p_{h\varphi_{non}t+1} = 0 \text{ if } \left(\frac{r^* + \theta}{1-\theta}\right)\acute{\rho}(f_t) = \frac{\Omega_{nont}}{(1-\theta)f_t^{\Pi}} + q\left[\rho\left(f_{t+1}\right) - \acute{\rho}(f_t)\right]$$

$$\Delta f_{\varphi_{non}t+1} = 0 \text{ if } \grave{\rho}(f_t) = \frac{\Omega_{nont}}{(1+r^*)f_t^{\Pi}} + \left(\frac{1-\theta}{1+r^*}\right)\left(\frac{\theta}{\Gamma}\right)^{\frac{1}{\Psi}} f_t^{\frac{1}{\Psi}}$$

$$\Delta f_{\varphi_{cb}t+1} = 0 \text{ if } \breve{\rho}(f_t) = \frac{(1+r^*)W}{(1-\theta)f_t} + \left(\frac{1-\theta}{1+r^*}\right)\left(\frac{\theta}{\Gamma}\right)^{\frac{1}{\Psi}} f_t^{\frac{1}{\Psi}}$$

Define $\tilde{\rho}(f)$ below.

$$\hat{\rho}(f) = \begin{cases} \acute{\rho}(f) & \text{for} \quad f \in (0, \bar{f}_{non}) \\ \grave{\rho}(f) & \text{for} \quad f \in [\bar{f}_{non}, \hat{f}) \\ \breve{\rho}(f) & \text{for} \quad f \in [\hat{f}, \infty) \end{cases}$$

[34] Note that Claim 8 is used in Figure 3a to capture the topological feature of all the loci.

By Claim 4 $\grave{\rho}(\hat{f}) = \breve{\rho}(\hat{f})$. Consequently, Figure 11 follows. Figure 11 shows the directional field over (p_h, f) for each area[35].

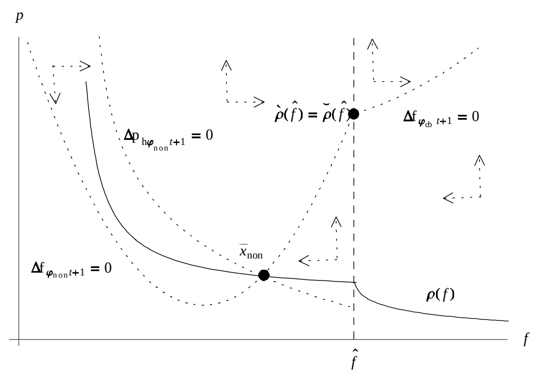

Figure 11: Derivation

Trivially from Figure 11, property 1 and 2 in Propostion 7 are satisfied. In particular, for $p_{ht} > \breve{\rho}(f_t)$ where $f_t \in [\hat{f}, \infty)$, $p_{ht+1} > p_{ht}$ and $f_{t+1} > f_t$. For $p_{ht} > \max\{\acute{\rho}(f_t), \grave{\rho}(f_t)\}$ where $f_t \in [0, \hat{f})$, $p_{ht+1} > p_{ht}$, $f_{t+1} > f_t$, and $p_{ht+1} > \max\{\acute{\rho}(f_{t+1}), \grave{\rho}(f_{t+1})\}$. Since $\grave{\rho}(f_t) > \breve{\rho}(f_t)$ for $f_t \in [\hat{f}, \infty)$, property 3 is proved from the figure. Property 4 follows the known result that the dynamics will not fall below $\rho(f)$ and from the directional field the dynamics will break through $\hat{\rho}(f)$ at some point in time. ■

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