SELECTED APPLICATIONS OF LOGIC TO C*-ALGEBRAS

ILIJAS FARAH

During the last decade the interface between set theory and operator algebras advanced from virtually nonexistent to a lively area of research. I will focus my attention to two rather different topics in this area.

Concrete C*-algebra is a norm-closed, self-adjoint algebra of bounded operators on a complex Hilbert space. An abstract C*-algebra is a Banach algebra with involution that is isomorphic to a concrete C*-algebra. Essentially by a 1942 result of Gelfand–Naimark–Segal, C*-algebras are axiomatizable in the logic of metric structures ([1], see [10]). The category of abelian C*-algebras is equivalent to the category of locally compact Hausdorff spaces, and therefore the study of C*-algebras is sometimes called 'noncommutative' or 'quantized' topology (cf. [3])

0.1. **Classification of nuclear, separable C*-algebras.** Elliott's program of classification of separable nuclear C*-algebras by K-theoretic invariants has enjoyed tremendous success and achieved a number of spectacular results ([19]). However, counterexamples constructed by Rørdam and Toms have shown that the program in its original formulation needs to be revised (see [6]).

Set-theoretic analysis of the Elliott program was initiated in [12] and [11] as a continuation of the broad endeavour to study the comparative complexity of classification problems (see [15] and [14]). I will present the current (rapidly changing!) state of the art on this subject.

0.2. **Rigidity of corona algebras.** The *multiplier algebra* $M(A)$ of a C*-algebra A is the non-commutative analogue of the Čech–Stone compactification and it is the maximal C*-algebra that has A as an essential ideal. The quotient $M(A)/A$ is called *corona algebra* (or *outer multiplier algebra*) and is a noncommutative analogue of the Čech–Stone remainder of a topological space. *Calkin algebra* is the simplest non-commutative corona and the simplest non-commutative instance of the rigidity problem for corona algebras is the 1977 question of Brown, Douglas and Fillmore whether Calkin algebra has outer automorphisms. By [18] and [8], the answer to this question is independent from ZFC. I will present known results and open problems on this question (see [9], [4]).

0.3. **References.** Only the basic knowledge of descriptive set theory and functional analysis will be assumed. Prerequisites for §0.1 and §0.2 roughly correspond to the prerequisites for [12] and [8], respectively. I shall nevertheless list some suggested readings. For the general theory of C*-algebras see [2], [17], or [5]. For their set-theoretic aspects see [20] and [13]. An excellent reference for descriptive set theory is [16], see also [14]. In §0.2 I will use model theory of metric structures ([1], [10]), combinatorial set theory, and what not (see [8], [7]).

References

[1] I. Ben Yaacov, A. Berenstein, C.W. Henson, and A. Usvyatsov. Model theory for metric structures. In Z. Chatzidakis et al., editors, *Model Theory with Applications to Algebra and Analysis, Vol. II*, number 350 in London Math. Soc. Lecture Notes Series, pages 315–427. Cambridge University Press, 2008.

[2] B. Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[3] A. Connes. *Noncommutative geometry*. Academic Press, 1994.

[4] S. Coskey and I. Farah. Group cohomology and corona automorphisms. preprint, 2012.

[5] K.R. Davidson. *C^*-algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.

[6] G.A. Elliott and A.S. Toms. Regularity properties in the classification program for separable amenable C^*-algebras. *Bull. Amer. Math. Soc. (N.S.)*, 45(2):229–245, 2008.

[7] I. Farah. *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*. Number 702 in Memoirs of the American Mathematical Society, vol. 148. 2000.

[8] I. Farah. All automorphisms of the Calkin algebra are inner. *Annals of Mathematics*, 173:619–661, 2011.

[9] I. Farah and B. Hart. Countable saturation of corona algebras. preprint, arXiv:1112.3898v1, 2011.

[10] I. Farah, B. Hart, and D. Sherman. Model theory of operator algebras II: Model theory. preprint, arXiv:1004.0741, 2010.

[11] I. Farah, A.S. Toms, and A. Törnquist. The descriptive set theory of C*-algebra invariants. preprint.

[12] I. Farah, A.S. Toms, and A. Törnquist. Turbulence, orbit equivalence, and the classification of nuclear C*-algebras. *J. Reine Angew. Math.*, to appear.

[13] I. Farah and E. Wofsey. Set theory and operator algebras. In J. Cummings and E. Schimmerling, editors, *Appalachian set theory 2006-2010*. Cambridge University Press, to appear.

[14] Su Gao. *Invariant descriptive set theory*, volume 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009.

[15] G. Hjorth. *Classification and orbit equivalence relations*, volume 75 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2000.

[16] A.S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate texts in mathematics*. Springer, 1995.

[17] Gerard J. Murphy. *C^*-algebras and operator theory*. Academic Press Inc., Boston, MA, 1990.

[18] N.C. Phillips and N. Weaver. The Calkin algebra has outer automorphisms. *Duke Math. Journal*, 139:185–202, 2007.

[19] M. Rørdam. *Classification of nuclear C^*-algebras*, volume 126 of *Encyclopaedia of Math. Sciences*. Springer-Verlag, Berlin, 2002.

[20] N. Weaver. Set theory and C^*-algebras. *Bull. Symb. Logic*, 13:1–20, 2007.