# Subcomplete Forcing and $\mathcal{L}$-Forcing

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April 7, 2012

ABSRACT

In his book **Proper Forcing** (1982) Shelah introduced three classes of forcings (complete, proper, and semi-proper) and proved a strong iteration theorem for each of them: The first two are closed under countable support iterations. The latter is closed under revised countable support iterations subject to certain standard restraints. These theorems have been heavily used in modern set theory. For instance using them, one can formulate "forcing axioms" and prove them consistent relative to a supercompact cardinal. Examples are PFA, which says that Martin's axiom holds for proper forcings, and MM, which says the same for semiproper forcings. Both these axioms imply the negation of CH. This is due to the fact that some proper forcings add new reals. Complete forcings, on the other hand, not only add no reals, but also no countable sets of ordinals. Hence they cannot change a cofinality to $\omega$. Thus none of these theories enable us e.g. to show, assuming CH, that Namba forcing can be iterated without adding new reals.

More recently we discovered that the three forcing classes mentioned above have natural generalizations which we call "subcomplete", "subproper" and "semi-subproper". It turns out that each of these is closed under Revised Countable Support (RCS) iterations subject to the usual restraints.

The first part of our lecture deals with subcomplete forcings. These forcings do not add reals. Included among them, however, are Namba forcing, Prikry forcing, and many other forcings which change cofinalities. This gives a positive solution to the above mentioned iteration problem for Namba forcing. Using the iteration theorem one can also show that the **Subcomplete Forcing Axiom** (SCFA) is consistent relative to a supercompact cardinal. It has some of the more striking consequences of MM but is compatible with CH (and in fact with $\diamond$).

(Note: Shelah was able to solve the above mentioned iteration problem for Namba forcing by using his ingenious and complex theory of "I-condition forcing". The relationship of I-condition forcing to subcomplete forcing remains a mystery. There are, however, many applications of subcomplete forcing which have not been replicated by I-condition forcing.)

In the second part of the lecture, we give an introduction to the theory of "$\mathcal{L}$-Forcings". We initially developed this theory more than twenty years ago in order to force the existence of new reals. More recently, we discovered that there is an interesting theory of $\mathcal{L}$-Forcings which do **not** add reals. (In fact, if we assume CH $+ 2^{\omega_1} = \omega_2$, then Namba forcing is among them.) Increasingly we came to feel that there should be a "natural" iteration theorem which would apply to a large class of these forcings. This led to the iteration theorem for subcomplete forcing.

Combining all our methods, we were then able to prove:

(1) Let $\kappa$ be a strongly inaccessible cardinal. Assume CH. There is a subcomplete forcing extension in which $\kappa$ becomes $\omega_2$ and every regular cardinal $\tau \in (\omega_1, \kappa)$ acquires cofinality $\omega$.

(2) Let $\kappa$ be as above, where GCH holds below $\kappa$. Let $A \subset \kappa$. There is a subcomplete forcing extension in which:

   - $\kappa$ becomes $\omega_2$;
   - If $\tau \in (\omega_1, \kappa) \cap A$ is regular, then it acquires cofinality $\omega$;
   - If $\tau \in (\omega_1, \kappa) \setminus A$ is regular, then it acquires cofinality $\omega_1$.

We will not be able to fully prove these theorems in our lectures, but we hope to develop some of the basic methods involved.