Moments and Positive Polynomials for Optimization III: PUTINAR VERSUS KARUSH-KUHN-TUCKER

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Global Optimization

Recall the GLOBAL optimization problem P:

$$f^* := \min_x \{ f(x) \mid g_j(x) \geq 0, j = 1, \ldots, m \},$$

where $f, g_j \in \mathbb{R}[X]$. Hence, the feasible set

$$\mathbf{K} := \{ x \in \mathbb{R}^n \mid g_j(x) \geq 0, j = 1, \ldots, m \}$$

is a basic semi-algebraic set.

Putinar versus Karush-Kuhn-Tucker

Let $f^* := \min_x \{ f(x) \, : \, g_j(x) \geq 0, \, j = 1, \ldots, m\}$ and let $x^* \in \mathbf{K}$ be a minimizer at a LOCAL minimum.

Karush-Kuhn-Tucker (KKT) OPTIMALITY CONDITIONS

There exist NONNEGATIVE SCALAR MULTIPLIERS $\lambda \in \mathbb{R}^m$ such that:

$$\nabla [\, f(x^*) - \sum_{j=1}^m \lambda_j \, g_j(x^*) \,] \, = \, 0. \quad \lambda_j \, g_j(x^*) = 0; \quad \lambda_j \geq 0$$

Under some constraint qualifications:

I. The KKT-optimality conditions are necessary for x^* to be a LOCAL minimizer only.

II. If f and $-g_j$ are concave, the KKT-optimality conditions are also sufficient for x^* to be a GLOBAL minimizer.

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IN GENERAL, x^* IS NOT a global minimizer of the LAGRANGIAN

$$x \mapsto L(x) \;:=\; f(x) - f^* - \sum_{j=1}^{m} \lambda_j\, g_j(x)$$

but ONLY a stationary point!

However, in the CONVEX case

x^* is a global minimizer of the Lagrangian L and:

$$L \geq 0 \quad \text{on } \mathbb{R}^n; \qquad L(x^*) = 0; \qquad \nabla L(x^*) = 0$$

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Putinar's representation theorem (Positivstellensatz)

$$f(x) \;=\; \sigma_0(x) + \sum_{j=1}^{m} \sigma_j(x)\, g_j(x), \qquad \forall x \in \mathbb{R}^n,$$

(for some s.o.s. polynomials (σ_j)) ....

holds for polynomials f that are STRICTLY POSITIVE on **K**.

However, by recent results from Marshall (2009), Nie (2012)

it also holds GENERICALLY for polynomials f that are only NONNEGATIVE on **K**!

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If **Putinar's** Theorem holds for $f - f^*$ (only ≥ 0 on $\mathbf{K}$), then

the EXTENDED LAGRANGIAN polynomial

$$x \mapsto \Psi(x) := f(x) - f^* - \sum_{j=1}^m \sigma_j(x)\, g_j(x) \qquad (= \sigma_0(x))$$

(with s.o.s. MULTIPLIERS $\sigma_j \in \mathbb{R}[X]$ instead of scalar $\lambda \in \mathbb{R}^m$)

is s.o.s.! (hence $\Psi \geq 0$ on $\mathbb{R}^n$), and satisfies:

$$\nabla\Psi(x^*) \;=\; \nabla f(x^*) - \sum_{j=1}^m \underbrace{\sigma_j(x^*)}_{\lambda_j^* \geq 0} \nabla g_j(x^*)\,] = 0$$

$$\sigma_j(x^*)\, g_j(x^*) \;=\; 0 \quad \forall j \qquad (\text{and so } \Psi(x^*) = 0)$$

$\rightarrow (x^*, \lambda^*)$ is a KKT pair.

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That is ...

x^* is a GLOBAL MINIMIZER
of the EXTENDED LAGRANGIAN Ψ on $\mathbb{R}^n$!

So when Putinar's representation holds

for the polynomial $f - f^*$ (which is only nonnegative on $\mathbf{K}$)

it provides a **global optimality certificate** for f^* and $x^* \in \mathbf{K}$

... the analogue

in **nonconvex polynomial optimization**

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An important property

On non active constraints

Let $(x^*, \lambda) \in \mathbf{K} \times \mathbb{R}^m_+$ be a KKT point with x^* a global minimizer of **P** and suppose that the constraint $g_j \geq 0$ is not active at x^*, i.e., $g_j(x^*) > 0$.

Then,

in contrast to KKT optimality conditions where the associated scalar multiplier λ_j vanishes ($\lambda_j = 0$), ...

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Indeed, even if NOT ACTIVE at x^*,

IN THE NONCONVEX case, the constraint $g_j(x) \geq 0$ MAY STILL BE IMPORTANT because if deleted, the global optimum f^* may strictly decrease to $\theta < f^*$.

Therefore in the case where $\theta < f^*$

the constraint $g_j(x) \geq 0$ MUST PLAY a ROLE in Putinar's representation of the polynomial $f - f^*$, i.e., its associated s.o.s. weight σ_j is NOT trivial.

Otherwise if $\sigma_j = 0$, i.e., if $f - f^* = \sigma_0 + \sum_{k \neq j} \sigma_k \, g_k$ then

$$\theta \,=\, \min_x \{f(x) \,:\, g_k(x) \geq 0, \,\forall\, k \neq j\} \,=\, f^*.$$

However, its VALUE at x^* VANISHES ($\sigma_j(x^*) = 0$)!

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Example: Consider the one-dimensional problem

$$\mathbf{P}: \quad f^* = \min_x \{-x \mid x^2 = 1;\ 1/2 - x \geq 0\},$$

with $X \mapsto g_1(X) = X^2 - 1$ and $X \mapsto g_2(X) := 0.5 - X$.

$x^* = -1$ is a global minimizer with global minimum $f^* = 1$.

$(x^*, \lambda) = (-1, (1/2, 0))$ is a KKT pair, and $\lambda_2 = 0$ because the constraint $g_2(x) \geq 0$ is not active at $x^* = -1$.

Of course, x^* is not a global minimum of the Lagrangian $f - \lambda_1 g_1 - \lambda_2 g_2 = -X - 1/2(X^2 - 1) = -X^2/2 - X + 1/2$.

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But we also have Putinar's representation

$$f - f^* = -X - 1 = (X + 3/2)(X^2 - 1) + (X + 1)^2(1/2 - X).$$

The s.o.s. (polynomial) multiplier $x \mapsto \sigma_2(X) := (X + 1)^2$ vanishes at $x^* = -1$, also a global minimizer of the Lagrangian

$$f - \sigma_1\, g_1 - \sigma_2\, g_2 = -X - (X + 3/2)(X^2 - 1) - (X + 1)^2(1/2 - X)$$

(here constant $\equiv 1$).

Even if not active at x^*, the constraint $g_2(x) \geq 0$ is important because if deleted, $f^* \to -1 < 1$. Therefore, it MUST have a nontrivial s.o.s. multiplier in the representation of $f - f^*$.

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