Signal reconstruction from the magnitude of subspace components

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Outline

- Problem formulation
- Configurations allowing for closed formulas
- Constructions
- SDP reconstruction for random configurations

Problem formulation

- Given a set of k-dimensional subspaces of $\mathbb{R}^d$:

$$\{V_j\}_{j=1}^n \qquad V_j \subset \mathbb{R}^d,\ \dim(V_j) = k$$

- Given $x \in \mathbb{R}^d$, $\|x\| = 1$, can we recover x (up to sign) from

$$\|P_{V_1}(x)\|, \ldots, \|P_{V_n}(x)\| \ ?$$

- Aims: exact recovery versus probabilistic recovery.

Goals

- Exact recovery from a closed formula. Requires special properties of $\{V_1, \ldots, V_n\}$ that nicely relate to cubature formulas, Grassmann designs, group representation.

  Requires $n \geq cd^2$.

  Extends and strengthens Balan, Bodmann, Casazza, Edidin (2009) for $k = 1$

- Recovery with high probability, using semidefinite programming, under a random choice of $\{V_1, \ldots, V_n\}$.

  Requires only $n \geq cd \log(d)$.

  Follows the lines of Candès, Strohmer, Voroninski (2011) for $k = 1$.
  Candes, Li (2012): $n \geq c\log(d)$ is enough.

Grassmann space

- $\mathcal{G}_{k,d}$ the space of subspaces of $\mathbb{R}^d$ of dimension k.
- The orthogonal group $O(\mathbb{R}^d)$ acts transitively on $\mathcal{G}_{k,d}$ and induces an invariant probability measure σ_k.
- Polynomial functions $\mathrm{Pol}_{\leq 2p}(\mathcal{G}_{k,d})$.

Definition: A cubature formula of strength $2p$ is a set $\{(V_j, \omega_j)\}_{j=1}^n$ such that $\omega_j > 0$, $\sum_j \omega_j = 1$, and

$$\int_{\mathcal{G}_{k,d}} f(V) d\sigma_k(V) = \sum_{j=1}^{n} \omega_j f(V_j) \qquad \text{for all } f \in \mathrm{Pol}_{\leq 2p}(\mathcal{G}_{k,d}).$$

$\omega_j = 1/n$: Grassmannian designs (of strength $2p$) (B., Coulangeon, Nebe 2002)

Exact recovery

Theorem: (B., Ehler) Let $\{(V_j, \omega_j)\}_{j=1}^n$ be a cubature formula of strength 4. Then,

$$P_x = \frac{1}{\alpha}\sum_{i=1}^{n} \omega_j \|P_{V_j}(x)\|^2 P_{V_j} - \frac{\beta}{\alpha} I_d$$

where $\alpha = \frac{2k(d-k)}{d(d+2)(d-1)}$ and $\beta = \frac{k(kd+k-2)}{d(d+2)(d-1)}$.

Sketch of proof:

$V \to \langle P_x, P_V\rangle\langle P_y, P_V\rangle \in \mathrm{Pol}_{\leq 4}(\mathcal{G}_{k,d})$ so:

$$\int_{\mathcal{G}_{k,d}} \langle P_x, P_V\rangle\langle P_y, P_V\rangle d\sigma_k(V) = \sum_{j=1}^{n} \omega_j \langle P_x, P_{V_j}\rangle\langle P_y, P_{V_j}\rangle.$$

$$\text{Let} \qquad K(x,y) := \int_{\mathcal{G}_{k,d}} \langle P_x, P_V\rangle \langle P_y, P_V\rangle d\sigma_k(V).$$

By $O(\mathbb{R}^d)$-invariance of σ_k, also $K(x,y)$ is $O(\mathbb{R}^d)$-invariant. It shows that, for some constants α, β,

$$K(x,y) = \alpha(x^* y)^2 + \beta.$$

Leads to:

$$\sum_{j=1}^{n} \underbrace{\omega_j \langle P_x, P_{V_j}\rangle \langle P_y, P_{V_j}\rangle}_{\langle \omega_j \|P_{V_j}(x)\|^2 P_{V_j}, P_y\rangle} = \underbrace{\alpha(x^* y)^2 + \beta}_{\langle \alpha P_x + \beta I, P_y\rangle}$$

It remains to compute α and β (easy).

Exact recovery in presence of erasures

It is sometimes possible to recover $\{\|P_{V_j}(x)\|\}_{j=1}^n$ even if p of these values are missing. It is the case when $\{(V_j, \omega_j)\}_{j=1}^n$ is a tight p-fusion frame.

Definition: A tight p-fusion frame is a set $\{(V_j, \omega_j)\}_{j=1}^n$ such that $\omega_j > 0$, $\sum_j \omega_j = 1$, and

$$\int_{\mathcal{G}_{k,d}} f(V) d\sigma_k(V) = \sum_{j=1}^n \omega_j f(V_j) \qquad \text{for all } f \in \mathrm{Pol}^1_{\leq 2p}(\mathcal{G}_{k,d}).$$

where $\mathrm{Pol}^1_{\leq 2p}(\mathcal{G}_{k,d}) \subset \mathrm{Pol}_{\leq 2p}(\mathcal{G}_{k,d})$ is the subspace 'generated' by $\mathrm{Pol}_{\leq 2p}(\mathcal{G}_{1,d})$.

B., Ehler, *Tight p-fusion frames* (2011), arXiv:1201.1798

Tight p-fusion frames

Equivalent characterizations:

1. $\{(V_j, \omega_j)\}_{j=1}^n$ is a tight p-fusion frame
2. There exists a constant A_p such that, for all $x \in S^{d-1}$,

$$\sum_{j=1}^{n} \omega_j \|P_{V_j}(x)\|^{2p} = A_p.$$

3. For all $k = 1, \ldots, p$,

$$\sum_{i,j} \omega_i \omega_j P_{(2k)}(\underline{y}(V_i, V_j)) = 0,$$

where $P_{(2k)}(y_1, \ldots, y_k)$ are certain multivariate Jacobi polynomials attached to $\mathrm{Pol}^1_{\leq 2p}(\mathcal{G}_{k,d})$ and $\underline{y}(V_i, V_j) \in [0,1]^k$ are the squared cosine of the principal angles between V_i and V_j.

Tight p-fusion frames

- The constant in 2. can take only the value:

$$A_p = \frac{(k/2)_p}{(d/2)_p} \qquad (a)_p := a(a+1)\dots(a+p-1)$$

- From 3. we see that a tight p-fusion frame is also a tight ℓ-fusion frame for $\ell < p$. So, we have

$$\sum_{j=1}^{n} \omega_j \|P_{V_j}(x)\|^{2\ell} = A_\ell \qquad 1 \le \ell \le p$$

- Extends the notion of tight frames ($k = 1$, $p = 1$) and tight fusion frames ($k \ge 2$, $p = 1$).

Tight p-fusion frames correct p erasures

If the values $\|P_{V_1}(x)\|, \dots, \|P_{V_p}(x)\|$ are missing, they can be recomputed by solving the following system of algebraic equations:

$$\begin{cases} \omega_1 T_1 + \cdots + \omega_p T_p = A_1 - \sum_{j=p+1}^{n} \omega_j \|P_{V_j}(x)\|^2 \\ \omega_1 T_1^2 + \cdots + \omega_p T_p^2 = A_2 - \sum_{j=p+1}^{n} \omega_j \|P_{V_j}(x)\|^4 \\ \qquad \cdots\cdots \\ \omega_1 T_1^p + \cdots + \omega_p T_p^p = A_p - \sum_{j=p+1}^{n} \omega_j \|P_{V_j}(x)\|^{2p} \end{cases}$$

There are only finitely many solutions (at most $p!$. Think of $\omega_i = ct$). Moreover in the reconstruction process using

$$P_x = \frac{1}{\alpha} \sum_{i=1}^{n} \omega_j \|P_{V_j}(x)\|^2 P_{V_j} - \frac{\beta}{\alpha} I_d$$

it is likely that most solution will not give rise to a matrix of rank one. However it outputs a list of candidate signals x.

Existence and constructions

We want to adress the following questions:

- When do cubature formulas and tight p-fusion frames exist ?
- How can they be constructed ?

Existence

Theorem:

1. If $\{(V_j, \omega_j)\}_{j=1}^n$ is a cubature formula of strength $2p$, then

$$n \geq \dim(\mathrm{Pol}_{\leq p}(\mathcal{G}_{k,d})) \approx c_p d^p.$$

2. Such a configuration does exist, with number of elements

$$n \leq \dim(\mathrm{Pol}_{\leq 2p}(\mathcal{G}_{k,d})) \approx c'_p d^{2p}.$$

$p = 2$: $n \geq p(p+1)/2$ and $n \leq p^4/8$.
Standard results. See [de la Harpe, Pache 2005] for a general framework where Grassmann spaces fit.
Existence result 2. is non constructive (uses Caratheodory theorem).

Numerical constructions

Constructing a cubature formulas of strength say 4 amounts to solve an algebraic system of equations:

$$\begin{cases} \{(V_j, \omega_j)\}_{j=1}^n \\ \text{cub. str. 4} \end{cases} \iff \begin{cases} \sum_{j=1}^n \omega_j \varphi(V_j) = 0 \\ \text{for all } \varphi \in \mathrm{Pol}^0_{\leq 4}(\mathcal{G}_{k,d}) \end{cases}$$

$$\iff \begin{cases} \sum_{j=1}^n \omega_j \varphi_\ell(V_j) = 0 \\ \text{for all } \ell = 1, \dots, \dim(\mathrm{Pol}^0_{\leq 4}(\mathcal{G}_{k,d})) \end{cases}$$

where $\mathrm{Pol}^0_{\leq 4}(\mathcal{G}_{k,d}) := \{\varphi \in \mathrm{Pol}_{\leq 4}(\mathcal{G}_{k,d}) \ : \ \int \varphi d\sigma_k(V) = 0\}$ and $\{\varphi_1, \dots, \varphi_\ell, \dots\}$ is a basis of $\mathrm{Pol}^0_{\leq 4}(\mathcal{G}_{k,d})$.

Algebraic constructions

Constructions using symmetries:

- Let G be a finite subgroup of $O(\mathbb{R}^d)$. Can $\{(V_j, \omega_j)\}_{j=1}^n$ afford G as a transitive group of symmetries, and be a cubature of strength $2p$ (resp a tight p-fusion frames) ?
- If so, we can assume $\omega_j = 1/n$.
- Stronger condition: Can all the orbits of G on $\mathcal{G}_{k,d}$ be cubatures of strength $2p$ (resp tight p-fusion frames) ?

$$\text{Orbit of } V \quad : \quad G \cdot V := \{g(V) \ : \ g \in G\}$$

Constructions using symmetries

Theorem: The following are equivalent:

1. For all $2 \le k \le d/2$, for all $V \in \mathcal{G}_{k,d}$, $G \cdot V$ is a design of str. 4.
2. $(V_d^{(4)})^G = (V_d^{(2,2)})^G = (V_d^{(2)})^G = \{0\}$.

And also:

1. For all $k \le d/2$, for all $V \in \mathcal{G}_{k,d}$, $G \cdot V$ is a tight p-fusion frame.
2. $(\mathbb{R}[X_1, \dots, X_d]_{2p})^G = \mathbb{R}(X_1^2 + \cdots + X_d^2)^p$.

where, for a partition μ, V_d^μ denotes a specific irreducible representation of $O(\mathbb{R}^d)$ that occurs in $\mathrm{Pol}_{\le 2p}(\mathcal{G}_{k,d})$ iff $\ell(\mu) \le k$, $\deg(\mu) \le 2p$ and μ is even.

Constructions using symmetries

Examples:

1. Tiep 2006: complete classification of the groups that satisfy 1.
2. Designs of strength 4: Weyl groups of root systems A_2, D_4, E_6, E_7, E_8 (str. 6), $2.Co_1$ (str. 10).
3. An infinite family: the Clifford groups $\mathcal{C}_m$ acting on $\mathbb{R}^d$, $d = 2^m$ give rise to 6-designs.
   Conway. Hardin, Rains, Shor, Sloane (1999) in view of Grassmannian packings have described non-generic orbits: e.g. $k = 2^{m-1} = d/2$ and $n = d^2 + d - 2$.
4. With a modification of this construction using maximal spreads of isotropic subspaces in $\mathcal{F}_2^{2m}$, when $m - s$ divised m (e.g, $s = 0$), 4-designs in $\mathcal{G}_{2^s,2^m}$ with $n \approx (d-1)(d+2)/2$ can be constructed (B. 2004).

A construction using 'concatenation'

Ingredients: $k < \ell < d$

$$\{(V_j,\omega_j)\}_{j=1}^{n},\ V_j \in \mathcal{G}_{k,\ell} \qquad \{(W_i,\tau_i)\}_{i=1}^{m},\ W_i \in \mathcal{G}_{\ell,d}$$

Goal: cubature of str. 4 or tight p-ff in $\mathcal{G}_{k,d}$.

$f_i : \mathbb{R}^\ell \to W_i$ fixed isometries. $\qquad V_{ij} := f_i(V_j) \in \mathcal{G}_{k,d}$

Theorem: (B., Ehler 2011) If $\{(V_j,\omega_j)\}_{j=1}^{n}$ and $\{(W_i,\tau_i)\}_{i=1}^{m}$ are cubature formulas of strength 4, respectivement tight p-fusion frames, then

$$\{(V_{ij},\tau_i\omega_j)\}_{\substack{1\le i\le m\\ 1\le j\le n}}$$

is also a cubature formula of strength 4, respectivement a tight p-fusion frame.

Probabilistic recovery from trace minimization

- We follow the lines of: Candès, Strohmer, Voroninski: *PhaseLift: exact and stable signal recovery from magnitude measurements via convex programming* (2011) who deal with $k = 1$.
- Let $f_i := \langle P_x, P_{V_i} \rangle$. P_x is an optimal solution of:

$$(P_{\text{rank}}) \quad \min \left\{ \text{rank}(X) \ : \ X \succeq 0, \ \langle X, P_{V_i} \rangle = f_i \quad 1 \leq i \leq n \right\}$$

- As is standard, we replace rank minimization with trace minimization:

$$(P_{\text{trace}}) \quad \min \left\{ \text{trace}(X) \ : \ X \succeq 0, \ \langle X, P_{V_i} \rangle = f_i \quad 1 \leq i \leq n \right\}$$

- Question : under which conditions is P_x the unique optimal solution of (P')?

Probabilistic recovery from trace minimization

Theorem: (B., Ehler 2012, arxiv:1209.5986v1)
Let $x \in \mathbb{R}^d$, $\|x\| = 1$. There are constants $c_1, c_2 > 0$ such that, if $n \geq c_1 d$, and $\{V_j\}_{j=1}^n$ are chosen independently identically distributed according to σ_k, with probability at least $1 - e^{-c_2 n/d}$, P_x is the unique optimal solution of (P_{trace}).

For $k = 1$, Candès and Li, *Solving quadratic equations via PhaseLift where there are about as many equations as unknowns*, arxiv:1208.6247v2 have proved a stronger result: proba $1 - e^{-cd}$. As a consequence, the same result holds uniformly for all x.

An idea of the proof

Let

$$\begin{aligned} \mathcal{F} : S^d &\to \mathbb{R}^n \\ X &\mapsto \tfrac{d}{k}(\langle X, P_{V_i}\rangle)_{i=1}^n \end{aligned} \qquad \begin{aligned} \mathcal{F}^* : \mathbb{R}^n &\to S^d \\ y &\mapsto \tfrac{d}{k}\sum_{i=1}^n y_i P_{V_i} \end{aligned}$$

$$\begin{aligned} (P_{\text{trace}}) \quad & \min\left\{\langle X, I_d\rangle \ : \ X \succeq 0,\ \mathcal{F}X = f\right\} \\ &= \max\left\{y^* f \ : \ I_d - \mathcal{F}^*(y) \succeq 0\right\}. \end{aligned}$$

$$\langle X, I_d\rangle - y^* f = \langle X, I_d - \mathcal{F}^* y\rangle.$$

We assume $x = e = (1, 0, \ldots, 0)^*$. A dual certificate for unique optimality of ee^* is given by a matrix $Y \in \operatorname{rg}(\mathcal{F}^*)$ such that

An idea of the proof

$$Y = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & Y' & \\ 0 & & & \end{pmatrix} \quad \text{and} \quad I_d - Y' \succ 0$$

Remember the reconstruction formula for a design of strength 4:

$$P_e = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{\alpha}\|P_{V_j}(e)\|^2 - \frac{d\beta}{k\alpha}\right)P_{V_j}$$

The RHS is a valid dual certificate Y! Motivates to take in general

$$Y := \frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{\alpha}\|P_{V_j}(e)\|^2 - \frac{d\beta}{k\alpha}\right)P_{V_j}.$$

An idea of the proof

With high probability, under the assumptions of the theorem, this matrix satisfies:

1. $\|Y_T - P_e\| \leq \gamma$
2. $\|Y_{T^\perp}\|_\infty \leq 1/2$

and $\mathcal{F}$ satisfies and 'almost isometry' property:

1. For all $X \succeq 0$,

$$(1-r)\|X\|_1 \leq \frac{1}{n}\|\mathcal{F}(X)\|_{\ell_1} \leq (1+r)\|X\|_1$$

2. For all X symmetric of rank at most 2,

$$\frac{1}{n}\|\mathcal{F}(X)\|_{\ell_1} \geq u_k(1-r)\|X\|_\infty$$

where $0 < r, u_k, \gamma < 1$ satisfy certain inequalities.