Improved bounds on book crossing numbers of complete bipartite graphs via semidefinite programming

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Workshop large-scale conic optimization, IMS, NUS, November 22nd, 2012

Outline

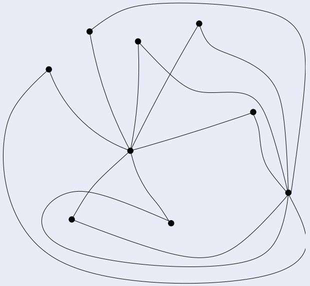
- The k-page crossing number of a complete bipartite graphs.
- Computing k-page crossing numbers by solving related max-k-cut problems.
- Semidefinite programming bounds and their implications.

Crossing number of a graph

Definition

The crossing number $\mathrm{cr}(G)$ of a graph $G = (V, E)$ is the minimum number of edge crossings that can be achieved in a drawing of G in the plane.

Example: the complete bipartite graph

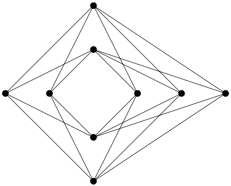


A drawing of $K_{2,7}$ with 2 edge crossings. Not optimal, since $\mathrm{cr}(K_{2,7}) = 0$.

The Zarankiewicz conjecture

$K_{m,n}$ can be drawn in the plane with at most $Z(m,n)$ edges crossing, where

$$Z(m,n) = \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor.$$



A drawing of $K_{4,5}$ with $Z(4,5) = 8$ crossings.

Zarankiewicz conjecture (1954)

$$\mathrm{cr}(K_{m,n}) \stackrel{?}{=} Z(m,n).$$

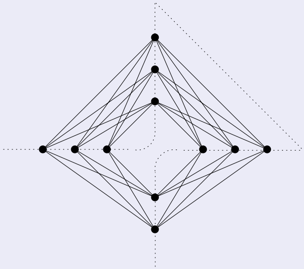
Known to be true for $\min\{m,n\} \leq 6$ (Kleitman, 1970), and some special cases.

k-page crossing number of a graph

Definition

In a k-page (book) drawing of $G = (V, E)$ all vertices V must be drawn on a straight line (the spine of a book), and each edge in one of k half-planes incident to this line (the book pages). The k-page crossing number $\nu_k(G)$ corresponds to k-page drawings of G.

Example: the complete bipartite graph $K_{5,6}$



"Straighten the dotted line" to get a two-page drawing.

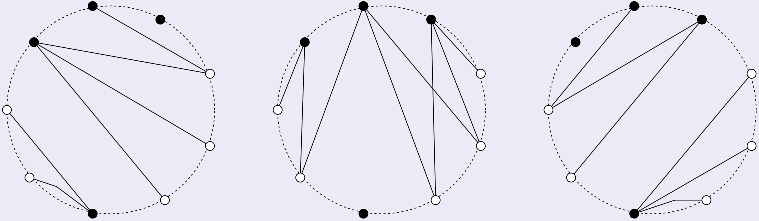
Equivalent k-page drawings

Circular drawings

In a circular drawing, the vertices are drawn on a circle, and all edges inside the circle.

A k-page drawing is equivalent to k circular drawings.

Example: A 3-page drawing of $K_{4,5}$ with 1 crossing.



Applications and complexity

- Crossing numbers are of interest for graph visualization, VLSI design, quantum dot cellular automata, ...
- It is NP-hard to compute $\mathrm{cr}(G)$ or $\nu_2(G)$ [Garey-Johnson (1982), Masuda et al. (1987)];
- The (k-page) crossing numbers of K_n and $K_{n,m}$ are not known in general, ...
- Crossing number of $K_{n,m}$ known as Turán brickyard problem — posed by Paul Turán in the 1940's.

Erdös and Guy (1973):

"Almost all questions that one can ask about crossing numbers remain unsolved."

Anno 2012 the situation has not changed much ... but there is some recent progress on lower bounds ...

Some known results

One page crossing number:

Theorem (Riskin (2003))

If $m \big| n$ *then* $\nu_1(K_{m,n}) = \frac{1}{12}n(m-1)(2mn-3m-n)$*, and this minimum value is attained when the* m *vertices are distributed evenly amongst the* n *vertices.*

Two page crossing number:

Theorem (De Klerk, Pasechnik, Schrijver (2007))

$$1 \geq \lim_{n\to\infty} \frac{\nu_2(K_{m,n})}{Z(m,n)} \geq \lim_{n\to\infty} \frac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \geq 0.8594 \textit{ if } m \geq 9.$$

Theorem (De Klerk and Pasechnik (2011))

$$\lim_{n\to\infty} \frac{\nu_2(K_{m,n})}{Z(m,n)} = 1 \textit{ if } m \in \{7,8\}.$$

These results use SDP lower bounds.

Some known results (ctd.)

k-**page crossing number:**

Theorem (Shahrokhi et al. 1996 (lower bound); De Klerk, Pasechnik, Salazar 2012 (upper bound))

$$\frac{1}{3(3\lceil \frac{k}{2} \rceil - 1)^2} \leq \lim_{m,n\to\infty} \frac{\nu_k(K_{m,n})}{\binom{m}{2}\binom{n}{2}} \leq \frac{1}{k^2}.$$

New results (this talk)

Theorem (De Klerk, Pasechnik, Salazar (2012))

Let $k \in \{2,3,4,5,6\}$, and let n be any positive integer. Define $\ell := \left\lfloor \frac{(k+1)^2}{4} \right\rfloor$ and $q := n \bmod \left\lfloor \frac{(k+1)^2}{4} \right\rfloor$. Then

$$\nu_k(K_{k+1,n}) = q \cdot \binom{\frac{n-q}{\ell}+1}{2} + (\ell - q) \cdot \binom{\frac{n-q}{\ell}}{2}.$$

Also, asymptotically ($k \to \infty$),

$$\lim_{k\to\infty} \left(\lim_{n\to\infty} \frac{\nu_k(K_{k+1,n})}{2n^2/k^2} \right) = 1.$$

New result: outline of the proofs

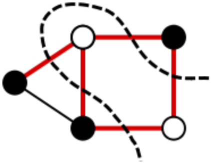
- We sketch the proof of the first result (lower bound only).
- One may compute $\nu_k(K_{m,n})$ by solving some maximum k-cut problems (Buchheim and Zheng (2007));
- We first show, for $2 \leq k \leq 6$, that $\nu_k(K_{k+1,s}) > 0$ for some $s = s(k)$.
- This is done by computing semidefinite programming bounds.
- Then we use a result by Turán to obtain lower bounds on $\nu_k(K_{k+1,n})$ for general n.

The maximum k-cut problem for graphs

Definition

For $G = (V, E)$ and a set of k colors, color V such that the number of edges with differently colored end points is a maximum. This maximum is denoted by max-k-cut(G).

Example ($k = 2$):



A maximum 2-cut example with max-k-cut$(G) = 5$.

$\nu_k(K_{m,n})$ and the maximum k-cut problem

$\nu_k(K_{m,n})$ may be obtained by solving some maximum k-cut problems ... one for each possible one-page (circle) drawing D of $K_{m,n}$.

Given D, the max-k-problem is solved for the graph $G_D(K_{m,n}) = (V_D, E_D)$ where:

- V_D is the edge set of $K_{n,m}$;
- Two vertices in V_D are adjacent if the corresponding edges cross in D.

Lemma (cf. Buchheim and Zheng (2007))

One has

$$\nu_k(K_{m,n}) = \min_D \left(|E_D| - \text{max-}k\text{-cut}(G_D(K_{n,m}))\right).$$

Proof: Given a k-page (circle) drawing of $K_{m,n}$, assign the edges drawn on page i the color i $(1 \leq i \leq k)$.

The number of one page drawings of $K_{m,n}$

Q: How may distinct one-page drawings are there of $K_{m,n}$?

A: as many as there are orbits of the dihedral group D_{m+n} acting on the set of one-page drawings.

Thus we may use the Burnside 'orbit counting' lemma.

Lemma (Orbit counting lemma (Frobenius?))

Let a finite group $\mathcal{G}$ act on a finite set Ω. Denote by Ω^g, for $g \in \mathcal{G}$, the set of elements of Ω fixed by g. Then the number N of orbits of $\mathcal{G}$ on Ω equals

$$N = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} |\Omega^g|.$$

The number of one page drawings of $K_{m,n}$

Applying the orbit counting lemma we obtain:

Lemma

Let $d = gcd(m, n)$. The number of distinct circular drawings of $K_{m,n}$ equals:

$$\frac{1}{2(m+n)}\begin{cases} \frac{m+n}{2}\left(\binom{\frac{m+n}{2}}{n/2} + \binom{\frac{m+n-2}{2}}{m/2} + \binom{\frac{m+n-2}{2}}{n/2}\right) + \sum_{k=0}^{d-1}\binom{\frac{m+n}{o(k)}}{\frac{m}{o(k)}} & (m,\ n \text{ even}) \\ (m+n)\binom{\frac{m+n-1}{2}}{n/2} + \sum_{k=0}^{d-1}\binom{\frac{m+n}{o(k)}}{\frac{m}{o(k)}} & (m \text{ odd},\ n \text{ even}) \\ (m+n)\binom{\frac{m+n-2}{2}}{(m-1)/2} + \sum_{k=0}^{d-1}\binom{\frac{m+n}{o(k)}}{\frac{m}{o(k)}} & (m,\ n \text{ odd}) \end{cases}$$

where $o(k)$ is the order of the subgroup generated by k in the additive group of integers $\mod d$.

Example: there are 1980 distinct one-page drawings of $K_{7,13}$.

Bounds from the ϑ-function

Lemma (Lovász (1979))

Given a graph $G = (V, E)$ and the value

$$\vartheta(G) := \max_{X \succeq 0} \left\{ \sum_{i,j \in V} X_{ij} \;\middle|\; X_{ij} = 0 \textit{ if } (i,j) \in E,\ \textit{trace}(X) = 1,\ X \in \mathbb{R}^{V \times V} \right\},$$

one has

$$\omega(\bar{G}) \le \vartheta(G) \le \chi(\bar{G}),$$

where $\omega(\bar{G})$ and $\chi(\bar{G})$ are the clique and chromatic numbers of the complement $\bar{G}$ of G, respectively.

Corollary

If $\vartheta(\overline{G_D(K_{m,n})}) > k$ for all one-page drawings D of $K_{m,n}$, then $\nu_k(K_{m,n}) > 0$.

Bounds from the ϑ-function (ctd)

We had:

Corollary

If $\vartheta(\overline{G_D(K_{m,n})}) > k$ for all one-page drawings D of $K_{m,n}$, then $\nu_k(K_{m,n}) > 0$.

By computing the ϑ-function for all distinct one-page drawings using DSDP (Benson-Ye) we obtained:

Theorem

For each $k \in \{2, 3, 4, 5, 6\}$,

$$\nu_k(K_{k+1,\lfloor (k+1)^2/4 \rfloor + 1}) > 0.$$

(For a few drawings, where $\vartheta(\overline{G_D(K_{m,n})}) = k$, we had to compute the chromatic number exactly using XPRESS-MP.)

We can use these results to obtain lower bounds on $\nu_k(K_{k+1,n})$ ($k \in \{2, 3, 4, 5, 6\}$) for general n, by using Turán's theorem.

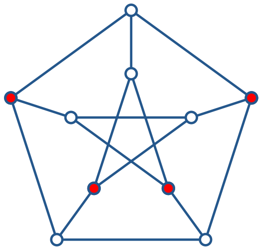
Turán's theorem

Theorem (Turán (1941))

If $G = (V, E)$ has coclique number $\alpha(G) \leq s$ for some s, then

$$|E| \geq \lceil (1/2)|V|^2(1/s - 1/|V|) \rceil.$$

Example: Petersen graph: $|V| = 10$, $|E| = 15$, $\alpha(G) = 4$.



$$|E| \geq \lceil (1/2)10^2(1/4 - 1/10) \rceil = 8.$$

Applying Turán's theorem

1. Assume $\nu_k(K_{k+1,s}) > 0$ for some k, s.
2. Let D be an optimal k-page drawing of $K_{k+1,n}$ for some $n > s$.
3. Now construct an auxiliary $G_D = (V_D, E_D)$: V_D is the vertices from the n co-clique of $K_{k+1,n}$. Two vertices in V are adjacent if two edges incident to them cross in D.
4. **NB:** $\nu_k(K_{k+1,n}) \geq |E_D|$ and $\alpha(G_D) \leq s$.
5. Now apply Turán's theorem to obtain:

$$\nu_k(K_{k+1,n}) \geq (1/2)n^2(4/(k+1)^2 - 1/n) \quad (2 \leq k \leq 6, n \geq (k+1)^2/4).$$

Conclusion and summary

- We demonstrated improved lower bounds on the k-page crossing numbers of some complete bipartite graphs.
- The proofs for small k were computer-assisted, using semidefinite programming (SDP) relaxations.
- Preprints available at arXiv.