Semidefinite Programming, Matrix Completion and Geometric Graph Realizations

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Large Scale Conic Optimization

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Goal of this lecture

Study conditions for the existence of
low rank positive semidefinite matrix completions
from a combinatorial point of view.

- Low rank solutions to semidefinite programs
- New graph parameter $\mathrm{gd}(G)$: *Gram dimension* of a graph

Geometric representations of graphs:

- Euclidean distance graph realizations
- Colin de Verdière type graph parameters

Geometry of SDP:

- Nondegeneracy: Unique completion, universal rigidity, SAP

Positive semidefinite matrix completion

Given a partial matrix:

$$\begin{pmatrix} \mathbf{1} & 0 & ? & -1 \\ 0 & \mathbf{1} & 1 & ? \\ ? & 1 & \mathbf{1} & 0 \\ -1 & ? & 0 & \mathbf{1} \end{pmatrix}$$

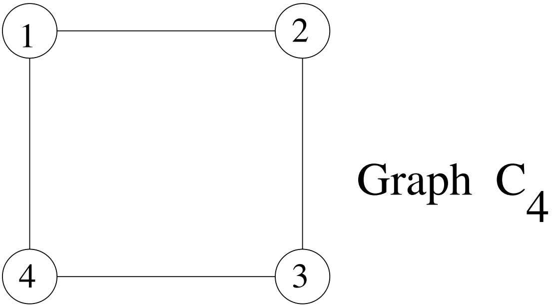
1. Can it be completed to a psd matrix?
2. How to find a psd completion?
3. Is such a completion unique?
4. What is the smallest rank of such a completion?

This lecture: Combinatorial approach

Given a partial matrix:

$$\begin{pmatrix} \mathbf{1} & 0 & ? & -1 \\ 0 & \mathbf{1} & 1 & ? \\ ? & 1 & \mathbf{1} & 0 \\ -1 & ? & 0 & \mathbf{1} \end{pmatrix}$$

Give answers depending on structural properties of the **graph of specified entries**:



Notation

- A symmetric matrix X is positive semidefinite (psd, $X \succeq 0$) if and only if it is a **Gram matrix**:

$$X = (u_i^\top u_j) \quad \text{for some vectors} \quad u_1, \ldots, u_n \in \mathbb{R}^k.$$

- $\mathcal{S}_+^n$ = all $n \times n$ psd matrices.

- Given a graph $G = (V = [n], E)$

  $\mathcal{S}_+(G)$ = all partial matrices $a \in \mathbb{R}^{V \cup E}$ (specified on the diagonal and on the edge set) that can be completed to a full psd matrix.

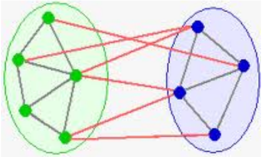
- The **elliptope** $\mathcal{E}_n$ = all psd matrices with an *all-ones diagonal* (set of **correlation matrices**).

- The projected elliptope $\mathcal{E}(G)$ = all partial matrices completable to a correlation matrix.

Low rank solutions to semidefinite programs

Why do we care about low rank solutions for SDP?

- **max-cut**= max $\sum_{ij \in E} w_{ij}(1 - \mathbf{x_i x_j})/2$ s.t. $\mathbf{x} \in \{\pm \mathbf{1}\}^{\mathbf{n}}$.
- sdp= max $\sum_{ij \in E} w_{ij}(1 - X_{ij})/2$ s.t. $X \succeq 0,\ X_{ii} = 1\ \forall i \in V$.

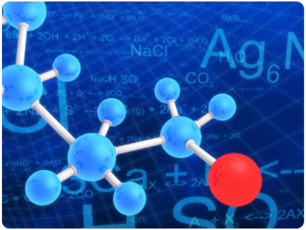


Let X be an optimal solution of the SDP.

1. **0.878**-approximation algorithm [Goemans-Williamson 1995].
2. If $\mathrm{rank}\ X \leq 4$, **0.881**-approximation [Avidor-Zwick 2005].
3. If $\mathrm{rank}\ X \leq 2$, **0.884**-approximation [Goemans]
4. If $\mathrm{rank}\ X = 1$, the SDP relaxation is **exact**: max-cut = sdp.

Another example: Distance geometry

Reconstruct the locations of objects (say) in 3D from partial measurements of mutual distances.



Find $u_1, \cdots, u_n \in \mathbb{R}^3$ such that $\|u_i - u_j\|^2 = d_{ij} \ \ \forall ij \in E$.

Equivalently: Find a solution of **rank at most 3** to the SDP:

$$X \succeq 0, \ X_{ii} + X_{jj} - 2X_{ij} = d_{ij} \ \ \forall ij \in E.$$

$\rightsquigarrow$ **Euclidean distance dimension** $\mathrm{ed}(G)$:
the smallest k s.t. there is a solution of rank at most k for any $d \in \mathbb{R}^E$

Getting low rank solutions via matrix completion

sdp$= \max\langle A_0, X\rangle$ s.t. $X \succeq 0,\ \langle A_k, X\rangle = b_k\ (k = 1, \cdots, m)$.

Sparsity pattern $G = (V, E)$: $ij \in E$ iff $(A_k)_{ij} \neq 0$ for some k.

Lemma

If X^* is an optimum solution, and X satisfies the system:

$$X_{ij} = X^*_{ij}\ \ (ij \in V \cup E),\quad X \succeq 0$$

then X too is an optimum solution.

Thus it suffices to find a low rank solution to the above **positive semidefinite matrix completion problem**.

The Gram dimension of a graph

Gram dimension

Definition

1. The **Gram dimension** $\mathrm{gd}(G, a)$ of a partial matrix $a \in \mathcal{S}_+(G)$ is the smallest rank of a psd completion of a.

   That is, the smallest k such that

   $$a_{ij} = u_i^\mathsf{T} u_j \quad \textit{for some} \quad u_1, \ldots, u_n \in \mathbb{R}^k.$$

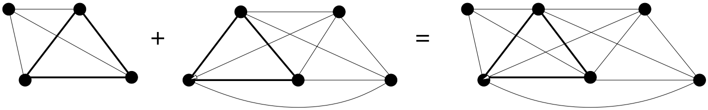
2. The **Gram dimension** $\mathrm{gd}(G)$ of a graph $G = (V, E)$ is

   $$\mathrm{gd}(G) = \max_{a \in \mathcal{S}_+(G)} \mathrm{gd}(G, a).$$

   That is, the smallest k such that any partial matrix $a \in \mathcal{S}_+(G)$ has a psd completion of rank at most k.

For instance, $\mathrm{gd}(K_n) = n$, $\quad \mathrm{gd}(G) \leq n - 1$ if $G \neq K_n$.

Case of chordal graphs



1. $a \in \mathcal{S}_+(G) \iff a[K] \succeq 0$ for all cliques K of G [Grone-Johnson-Sà-Wolkowicz 1984]
2. Compute a psd completion of $a \in \mathbb{Q}^{V \cup E}$ in poly-time [L 2000]
3. $\mathrm{gd}(G, a) = \max_K \mathrm{rank}\, a[K]$.
4. $\mathrm{gd}(G) = \max_K |K|$ for G chordal.

Theorem

For any graph G, $\mathrm{gd}(G) \leq tw(G) + 1$.

$\mathrm{tw}(G)$: **tree-width** of G = smallest integer k such that G is contained in a clique sum of cliques K_{k+1}

Application: Bounding ranks of solutions to SDP

Example (Göring, Helmberg, Reiss 2012: Minimizing the maximum eigenvalue of the weighted Laplacian)

$$\max_{\xi \in \mathbb{R}, v_1, \cdots, v_n \in \mathbb{R}^n} \left\{ \xi : \sum_{i \in V} \|v_i\|^2 = 1, \ \|v_i - v_j\|^2 \geq \xi \ (ij \in E) \right\}$$

has an optimal solution in dimension at most $\mathrm{gd}(G) \leq \mathrm{tw}(G) + 1$.

Example (Göring, Helmberg, Wappler 2008: Maximizing the second smallest eigenvalue of the weighted Laplacian)

$$\max_{v_1, \ldots, v_n \in \mathbb{R}^n} \left\{ \sum_{i \in V} \|v_i\|^2 : \|v_i - v_j\|^2 \leq 1 \ (ij \in E), \ \|\sum_{i \in V} v_i\|^2 = 0 \right\}$$

has an optimum solution in dimension at most $\mathrm{tw}(G) + 1$.

The bound $\mathrm{gd}(G) \leq \mathrm{tw}(G) + 1$ does not help in the second example, since the sparsity pattern is the complete graph.

Complexity of the Gram dimension parameter

Theorem (ELV 2012)

For any fixed $k \geq 2$, deciding whether $\mathrm{gd}(G, a) \leq k$ [i.e., a has a psd completion of rank at most k] is an **NP-hard problem.**

Membership in the **rank constrained elliptope** $\mathcal{E}_k(G)$ *is NP-hard.*

$\mathcal{E}_k(G)$=all partial matrices completable to a correlation matrix of rank at most k.

$$\mathcal{E}_k(G) \subseteq \mathrm{conv}\ \mathcal{E}_k(G) \subseteq \mathcal{E}(G).$$

Theorem (ELV 2012)

Membership in the convex hull of $\mathcal{E}_k(G)$ is NP-hard.

Question: Is weak optimization over $\mathrm{conv}(\mathcal{E}_k(G))$ NP-hard?

Case $k \geq 3$: Use orthogonal representations

$\mathrm{gd}(G, 0_E)$ *is the smallest integer* k *for which there exist unit vectors* $u_1, \cdots, u_n \in \mathbb{R}^k$ *such that* $u_i^\top u_j = 0 \;\; \forall ij \in E.$

Links to *cliques*, *graph colorings* and Lovász' *theta number* :

Theorem (Lovász' sandwich inequality)

$$\omega(G) \leq \vartheta(\overline{G}) \leq \mathrm{gd}(G, 0_E) \leq \chi(G).$$

- $\mathrm{gd}(G, 0_E) \leq 4$ for G planar graph.
- Deciding whether $\mathrm{gd}(G, 0_E) \leq 3$ is **NP-hard** for G planar. [Peeters 1997]
- For $k \geq 3$, deciding whether $\mathrm{gd}(G, 0_E) \leq k$ is **NP-hard** (for suspensions of planar graphs).
- For $k = 2$, $\mathrm{gd}(G, 0_E) \leq 2 \iff G$ is bipartite.

So we need another reduction!

Case $k = 2$: Use Euclidean graph realizations

Basic tool 1:

Characterize $\mathrm{ed}(G, d) \leq 1$ and $\mathrm{gd}(G, a) \leq 2$ in terms of a **partition type property** of the arguments d and $\theta = \arccos a$:

$$\exists \epsilon \in \{\pm 1\}^E \quad \sum_{e \in C} \epsilon_e d_e = 0, \quad \sum_{e \in C} \epsilon_e \theta_e \in 2\pi\mathbb{Z} \;\; \forall C \text{ (oriented) circuit}$$

Basic tool 2:

Hardness result of [Saxe 1979] for $\mathrm{ed}(G, d) \leq 1$ when $d \in \{1, 2\}^E$.

Structural characterizations

Gram and Euclidean graph realizations

Suspension graph: $\nabla G = G$ + new node adjacent to all nodes of G.

Theorem (LV 2012)

$\mathrm{gd}(G) \; = \; \mathrm{ed}(\nabla G) \; \geq \; \mathrm{ed}(G) + 1.$

Question

- *Does the inequality:* $\mathrm{ed}(\nabla G) \leq \mathrm{ed}(G) + 1$ *hold?*
- *Equivalently:* $\mathrm{gd}(G) \; = \; \mathrm{ed}(G) + 1$ *?*

Yes, if G has **Gram dimension at most 4**.

Forbidden minor characterization for $\mathrm{ed}(G) \leq 3$

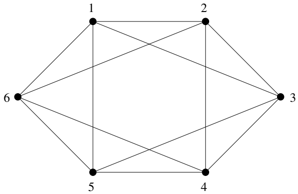
Theorem (Belk-Connelly 2007)

1. *The graph parameter* $\mathrm{ed}(G)$ *is* **minor monotone**:

$$\mathrm{ed}(G \setminus e), \mathrm{ed}(G/e) \leq \mathrm{ed}(G).$$

   Hence, for any k, the class of graphs with $\mathrm{ed}(G) \leq k$ can be characterized by **finitely many forbidden minors**.

2. $\mathrm{ed}(G) \leq 1 \iff G \text{ has no minor } K_3.$
3. $\mathrm{ed}(G) \leq 2 \iff G \text{ has no minor } K_4.$
4. $\mathrm{ed}(G) \leq 3 \iff G \text{ has no minor } K_5, K_{2,2,2}.$

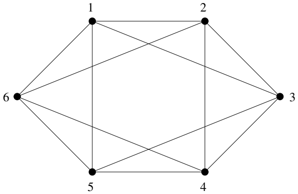


The octahedron graph $K_{2,2,2}$

Forbidden minor characterization for $\mathrm{gd}(G) \leq 4$

Theorem

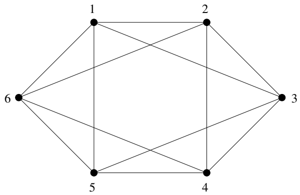
1. *The graph parameter* $\mathrm{gd}(G)$ *is* **minor monotone**.
2. $\mathrm{gd}(G) \leq 2 \iff G$ has no minor K_3.
3. $\mathrm{gd}(G) \leq 3 \iff G$ has no minor K_4.
4. $\mathrm{gd}(G) \leq 4 \iff G$ has no minor $K_5, K_{2,2,2}$.



Links between both forbidden minor characterizations

Theorem

1. *The graph parameter* $\mathrm{gd}(G)$ *is* **minor monotone**.
2. $\mathrm{gd}(G) \leq 2 \iff G$ has no minor $K_3 \implies \mathrm{ed}(G) \leq 1$.
3. $\mathrm{gd}(G) \leq 3 \iff G$ has no minor $K_4 \implies \mathrm{ed}(G) \leq 2$.
4. $\mathrm{gd}(G) \leq 4 \iff G$ has no minor $K_5, K_{2,2,2} \implies \mathrm{ed}(G) \leq 3$.



Recall:

$$\mathrm{ed}(G) \leq \mathrm{gd}(G) - 1.$$

Sketch of proof

Theorem (Arnborg, Proskurowski, Corneil 1990)

G has tree-width at most 3 $\iff$ no K_5, $K_{2,2,2}$, V_8, $C_5 \square K_2$ minor.

Theorem

$\mathrm{gd}(G) \leq 4 \iff G$ *has no* $K_5, K_{2,2,2}$ *minor.*

Sketch of proof:

1. K_5, $K_{2,2,2}$ have Gram dimension 5.
2. If G is 2-connected with no K_5, $K_{2,2,2}$ minor, then G is contained in a clique sum of copies of K_4, V_8, $C_5 \square K_2$.
3. V_8, $C_5 \square K_2$ have Gram dimension 4.

As in the work of Belk-Connelly [2007], the tedious part of the proof consists of showing that $\mathrm{gd}(V_8)$, $\mathrm{gd}(C_5 \square K_2) \leq 4$.

Following So-Ye [2007], use **SDP duality***: use the optimal dual (stress) matrix to 'fold' the optimal primal solution in low dim.*

Geometry of SDP and Colin de Verdière type graph parameters

How to show $\mathrm{gd}(G) \geq k$?

Recipe: Find a partial matrix a having a unique psd completion X and with $\mathrm{rank}\ X \geq k$. Consider the pair of primal and dual SDP's:

$$\sup_X \ 0 \ \text{ s.t. } X_{ij} = a_{ij} \ (ij \in V \cup E), \ X \succeq 0, \qquad (P_G)$$

$$\inf_{y,Z} \sum_{ij \in V \cup E} a_{ij} y_{ij} \ \text{ s.t. } Z = \sum_{ij \in V \cup E} y_{ij} E_{ij} \succeq 0. \qquad (D_G)$$

Theorem (fundamental facts about SDP)

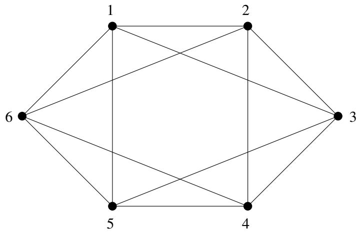
Let X be a completion of a, $X = Gram(p_1, \ldots, p_n)$ with $p_i \in \mathbb{R}^d$.

- *If there is a* **nondegenerate** *dual optimum solution Z, then X is the unique completion of a.*
- *Let Z be a dual optimal solution which is* **strictly complementary** *to X, i.e., corank $Z = \mathrm{rank}\ X$. TFAE:*
  1. *Z is dual nondegenerate.*
  2. *X is the unique psd completion of a.*
  3. *X is an extreme point of the primal feasible region:*

$$\{p_i^\mathsf{T} p_j : ij \in V \cup E\} \ \text{ spans } \mathcal{S}^d.$$

Example: $K_{2,2,2}$ has Gram dimension 5

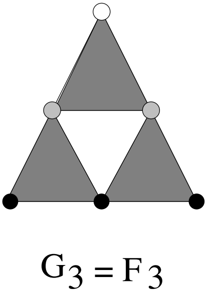
$K_{2,2,2} = K_6 \setminus \{14, 25, 36\}$

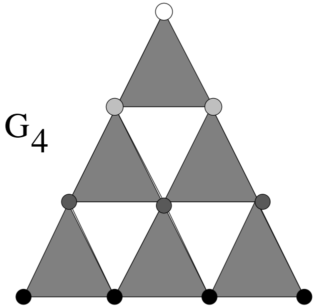


- $X = \mathrm{Gram}(e_1, e_2, e_3, e_4, e_5, \frac{e_1+e_2}{\sqrt{2}})$ is an extreme point of the primal feasible region, with rank 5.
- $a \in \mathcal{S}_+(K_{2,2,2})$ the corresponding partial matrix.
- $Z = (1, 1, 0, 0, 0, -\sqrt{2})(1, 1, 0, 0, 0, -\sqrt{2})^\mathsf{T}$ is dual optimal with corank 5.
- Hence: X is the **unique psd completion** of a.
- **This shows:** $\mathrm{gd}(K_{2,2,2}, a) = 5$.

Planar graphs with unbounded Gram dimension

The supertriangle G_r has $\mathrm{gd}(G_r) \geq r$.





Choose the vector labeling in such a way that each black triangle has rank 2.

The supertriangles are used in [Colin de Verdère 1998].

Dual nondegeneracy and the Strong Arnold Property

$$\sup_X \ 0 \ \text{ s.t. } X_{ij} = a_{ij} \ (ij \in V \cup E), \ X \succeq 0, \qquad (P_G)$$

$$\inf_{y,Z} \sum_{ij \in V \cup E} a_{ij} y_{ij} \ \text{ s.t. } Z = \sum_{ij \in V \cup E} y_{ij} E_{ij} \succeq 0. \qquad (D_G)$$

Definition

Let Z be dual feasible of rank r. Then, Z is dual **nondegenerate** if the tangent space $\mathcal{T}_Z$ to the manifold $\mathcal{M}_r$ (of rank r matrices) intersects transversally at Z the linear space:

$$\mathcal{L} = \mathrm{lin}\{E_{ij} : ij \in V \cup E\} = \{M : M_{ij} = 0 \ \forall ij \in \overline{G}\}.$$

That is,

$$ZR = 0, \ R_{ij} = 0 \ \forall ij \in V \cup E \Longrightarrow R = 0. \qquad \text{(SAP)}$$

SAP is known as the **Strong Arnold Property**. It is used to define Colin de Verdière type graph parameters $\mu(G), \nu(G), \nu_H(G), \ldots$.

The graph parameter $\nu_H(G)$

Definition (van der Holst 2003)

$\nu_H(G) = \max \ \text{corank}(Z) \ \text{s.t.} \ Z \succeq 0, \ Z_{ij} = 0 \ (ij \in \overline{E}), \ (SAP)$.

Theorem (van der Holst 2003)

1. *The parameter ν_H is minor monotone.*
2. $\nu_H(G) \leq 4 \iff$ *G has no minor K_5 or $K_{2,2,2}$.*

Same forbidden minors as for $\text{gd}(G) \leq 4$!

Theorem (LV 2012)

1. $\nu_H(G) \leq \text{gd}(G)$.
2. $\nu_H(G) = \max \text{gd}(G, a)$, *taken over all* **nice** *a, i.e., those for which the dual (D_G) has a nondegenerate optimal solution.*

Question

Does equality: $\text{gd}(G) = \nu_H(G)$ hold ?

Universal rigidity of frameworks

Definition

A framework $(G, \mathbf{p} = \{p_1, \ldots, p_n\})$ is **universally rigid** if for any $\mathbf{q}$:

$$[\|q_i - q_j\|^2 = \|p_i - p_j\|^2 \ \forall ij \in E] \Longrightarrow [\|q_i - q_j\|^2 = \|p_i - p_j\|^2 \ \forall i, j \in V]$$

Theorem (Connelly's sufficient conditions)

Let $p_1, \ldots, p_n \in \mathbb{R}^d$ which affinely span $\mathbb{R}^d$. Assume:

1. $R \in \mathcal{S}^d, \ \langle R, (p_i - p_j)(p_i - p_j)^\mathsf{T} \rangle = 0 \ \forall ij \in E \Longrightarrow R = 0.$
   [no conic at infinity]
2. *There is a psd* **stress matrix** *Z of corank d:*

$$Z_{ij} = 0 \ \forall ij \in \overline{E}, \ Ze = 0, \ \sum_{j \in V} Z_{ij} p_j = 0 \ \forall i \in V.$$

Then: (G, p) is universally rigid.

This extends to tensegrities.

Simple geometric proof

Let $q_1, \ldots, q_n$ such that $\|q_i - q_j\|^2 = \|p_i - p_j\|^2 \;\; \forall ij \in E$.
We need to show that $\|q_i - q_j\|^2 = \|p_i - p_j\|^2 \;\; \forall i,j \in V$.

1. Let $\widehat{p}_i = (p_i, 1)$ and $X = \text{Gram}(\widehat{p}_1, \ldots, \widehat{p}_n)$, $\mathrm{rank}\, X = d+1$.
   Let $Y = \text{Gram}(q_1, \ldots, q_n)$.
2. By the assumptions on Z: $ZX = 0 \Longrightarrow \ker X = \text{Range}(Z)$.
3. Moreover, $\langle Z, Y - X \rangle = 0 \Longrightarrow \langle Z, Y \rangle = \langle Z, X \rangle = 0$
   $\Longrightarrow ZY = 0 \Longrightarrow \ker Y \supseteq \ker X$.
4. Hence: $Y - X = (\langle R, \widehat{p}_i \widehat{p}_j^{\mathsf{T}} \rangle)_{i,j \in V}$ for some $R \in \mathcal{S}^{d+1}$.
5. The *no conic at infinity condition* implies: $R = \begin{pmatrix} \mathbf{0} & a \\ a^{\mathsf{T}} & b \end{pmatrix}$,
   where $a \in \mathbb{R}^d$, $b \in \mathbb{R}$.
6. Thus: $(Y - X)_{ij} = a^{\mathsf{T}} p_i + a^{\mathsf{T}} p_j + b \;\; \forall i,j \in V$, implying
   $\|q_i - q_j\|^2 = Y_{ii} + Y_{jj} - 2Y_{ij} = X_{ii} + X_{ij} - 2X_{ij} = \|p_i - p_j\|^2$.

Relevant preprints

• A new graph parameter related to bounded rank positive semidefinite matrix completions. With A. Varvitsiotis. arXiv:1204.0734, 2012.

• Complexity of the positive semidefinite matrix completion problem with a rank constraint. With M. E.-Nagy and A. Varvitsiotis. arXiv:1203.6602, 2012.

• Semidefinite programming, universal rigidity and the Strong Arnold Property. With A. Varvitsiotis. In preparation.

• On bounded rank positive semidefinite matrix completions of extreme partial correlation matrices. With M. E.-Nagy and A. Varvitsiotis. arXiv:1205.2040, 2012.