Inverse moment problems on polyhedra and optimization

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joint work with Nick Gravin (NTU/PDMI RAS), Jean B. Lasserre (LAAS), Sinai Robins (NTU), Boris Shapiro (Stockholm U.) and Michael Shapiro (Michigan State)

Outline

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Moment generating functions - the interval

$\mu_m^\rho := \int_P \mathbf{x}^m \rho(\mathbf{x}) d\mathbf{x}$ — the $\mathbb{Z}_+ \ni m$-th moment of ρ supported on $P \subset \mathbb{R}^d$. What is a "natural" generating function to encode μ_m^ρ's?

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Try Cauchy transform

$$F(u) = \int_P \frac{\rho(x)dx}{1 - \langle u, x\rangle} = \sum_{k\geq 0} \int_P \langle u, x\rangle^k \rho(x) dx = \sum_m \binom{|m|}{m} \mu_m^\rho u^m.$$

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$(u\frac{\partial}{\partial u} + 1) \circ F(u) = \int_a^b \frac{dx}{(1-ux)^2} = \sum_m (1+m)\mu_m u^m.$

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Fantappiè transformations, $\rho \equiv 1$

Let $F(u) = \int_P \frac{dx}{1-\langle u,x\rangle} = \sum_m \binom{|m|}{m} \mu_m u^m$, and $g(z) := g(z_1, \ldots, z_d) = \prod_{\ell=1}^{d} (\sum_k z_k + \ell)$.

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Definition

Fantappiè transformation ("rationalized" moment generating function) of $P \subset \mathbb{R}^d$ is

$$\mathcal{F}_P(u) := g(u_1\frac{\partial}{\partial u_1},\dots,u_d\frac{\partial}{\partial u_d}) \circ F(u)$$
$$= \int_P \frac{d!dx}{(1-\langle u,x\rangle)^{d+1}} = \sum_m \frac{(|m|+d)!}{\prod_j m_j!}\mu_m u^m.$$

Fantappiè transformation of a simplex

Example

Fantappiè transformation ("rationalized" moment generating function) of a simplex $\Delta = \mathrm{conv}(v_1, \ldots, v_{d+1}) \subset \mathbb{R}^d$ is

$$\mathcal{F}_\Delta(u) = \int_\Delta \frac{d!dx}{(1 - \langle u, x \rangle)^{d+1}} = \frac{d!\,\mathrm{Vol}(\Delta)}{\prod_k (1 - \langle v_k, u \rangle)}.$$

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This is known to be true in greater generality ($\mathbb{C}$-convexity), when Δ is a simplex in $\mathbb{C}P^{d-1}$. Cf. e.g. Andersson-Passare-Sigurdsson (2004).

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Corollary

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constant measures supp. on $\mathcal{P}(\mathbb{R}^d) \to$ rational functions on $\mathbb{R}^d$

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on the algbera $\mathcal{P}(\mathbb{R}^d)$ of polyhedra. Cf. Brion-Barvinok-Khovanskii-Lawrence -Pukhlikov

$$\mathcal{P}(\mathbb{R}^d) \to \text{meromorphic functions on } \mathbb{R}^d$$

$$P \mapsto \int_P e^{\langle u,x\rangle} dx$$

Formulae for Fantappiè transformations, $\rho \equiv 1$

Theorem

The Fantappiè transformation of a simple polytope $P \subset \mathbb{R}^d$ is

$$\mathcal{F}_P(u) = (-1)^d \sum_{v \in V(P)} \frac{\begin{vmatrix} w_1(v)-v \\ \cdots \\ w_d(v)-v \end{vmatrix}}{\prod_{j=1}^{d} \langle w_j(v), u \rangle} \cdot \frac{1}{1 - \langle v, u \rangle}$$

$$= (-1)^d \sum_{v \in V(P)} \langle v, u \rangle^d \begin{vmatrix} w_1(v)-v \\ \cdots \\ w_d(v)-v \end{vmatrix} \prod_{j=1}^{d} \langle w_j(v), u \rangle^{-1} \cdot \frac{1}{1 - \langle v, u \rangle}$$

where $w_1(v), \ldots, w_d(v)$ are generators of the vertex cone of $v \in V(P)$.

Vertices of compact non-convex polyhedra

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Definition

A *dissection* of P is a finite collection $\mathcal{T}$ of d-simplices $T_i = \mathrm{conv}(v_{i0}, \ldots, v_{id})$ s.t, $\mathrm{int}\, T_i \cap \mathrm{int}\, T_j = \emptyset$, $i \neq j$, and $P = \cup_i T_i$.

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Then we can take $V := V(P)$. (For P convex it's the usual vertices.) More details here: Gravin-DP-B.&M.Shapiro (2012)

Parametrizing P by simplices from $V(P)$

Theorem

Assume that any $d+2$ points in $V(P)$ affinely span $\mathbb{R}^d$ (non-degeneracy). *Then the constant unit density measure $[P]$ supported on P satisfies*

$$[P] = \sum_{\{v_0,\dots,v_d\} \subset V(P)} a_{v_0 \dots v_d} [\mathrm{conv}(v_0, \dots, v_d)]. \tag{1}$$

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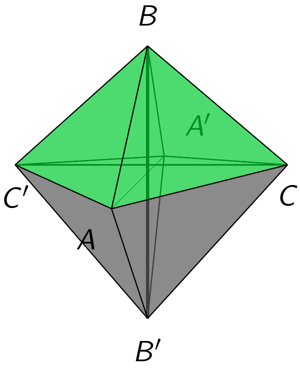

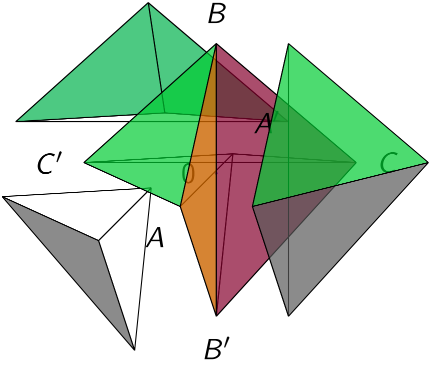

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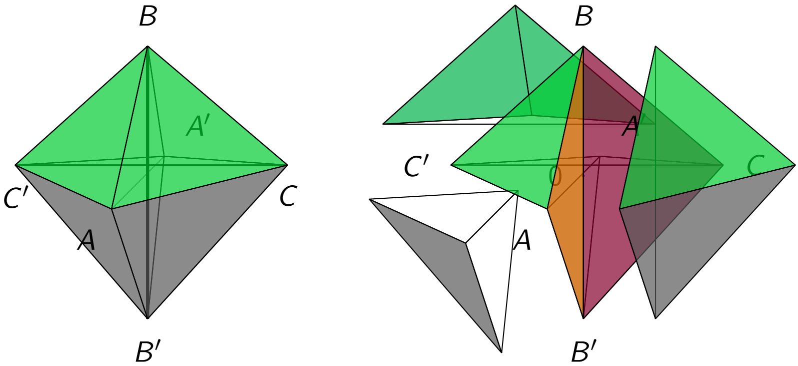

To dissect, it needs an extra vertex in the interior.

A glimpse of the proof

Apply Fantappiè transformation to both sides of (1)

$$\mathcal{F}_P(u) = \frac{P(u)}{\prod_{v\in P(V)}(1-\langle u,v\rangle)}$$

$$(???) \quad = \sum_{\Omega:=\{v_0,\dots,v_d\}\subset V(P)} a_\Omega \frac{\mathrm{Vol}(\mathrm{conv}(\Omega))}{\prod_{v\in\Omega}(1-\langle u,v\rangle)}.$$

We would be done if we manage to show that $P(u)$ lies in the vectorspace Π spanned by the products $\Pi_\Omega = \frac{\prod_{v\in P(V)}(1-\langle u,v\rangle)}{\prod_{v\in\Omega}(1-\langle u,v\rangle)}$, where $\Omega := \{v_0,\dots,v_d\} \subset V(P)$.
Non-degeneracy implies that Π_Ω, s.t. $v_0 \in \Omega$, span Π.
A bit of commutative algebra shows that $\dim \Pi$ coinsides with the dimnesion of full the space of polynomials in u of the appropriate degree, and the latter spanning set is a basis.

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Corollary

The Fantappiè transformation $\mathcal{F}_P^\rho(u)$ of an arbitrary (non-convex) polytope $\mathcal{P}$ w.r.t. an arbitrary homogeneous polynomaial weight ρ of degree δ is a rational function with denominator $\prod_{v\in V(P)}(1-\langle v,u\rangle)^\delta$.

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- specializing to univariate generating functions for axial moments Gravin-Lasserre-DP-Robins (2011)
- tackling $\mathcal{F}_P^\rho(u)$ directly

Outline

Axial moments

Specialize to axial moments: pick several $\mathbf{z} \in \mathbb{R}^d$, and consider

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Reconstruct P, by finding the kernel of the (infinite) Hankel matrix $H = (c_{i+j})_{0 \le i,j < \infty}$ from found, as denominators of $\Phi_{\mathbf{z}}(t)$, projections of $v \in \mathrm{Vert}(P)$ to $\mathbf{z}$. Use a variation of Prony method to solve the underlying nonlinear system.

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Let $\mathcal{F}_P(u) = \sum_I c_I u^I$, and $A = (c_{I+J})_{I,J}$, for a nice orderng of the monomial basis $B = \{1, u^{I_1}, u^{I_2}, \dots\}$.

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$c \in \operatorname{Ker} A$ iff for any u^K the Laurent series $u^{-K}\mathcal{F}_P(u)\sum_I c_I u^{-I}$ has no free term.

Corollary

Let $\sum_I c_I u^I = u^M q(\frac{1}{u_1}, \dots, \frac{1}{u_d}) \in \mathbb{R}[u]$, and u^M does not divide any monomial of $p(u)$. Then $c \in \operatorname{Ker} A$.

Fantappié moment matrices—questions

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Problem

Relate A and the "usual" moment matrices.

IMS Nov 2013/Jan 2014—announcement

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In particular a range of topics related to the present talk will be covered.