Towards higher order semidefinite relaxations for Max-Cut

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The Max-Cut Problem

Unconstrained quadratic 1/-1 optimization:

$$\max\ x^T L x \text{ such that } x \in \{-1, 1\}^n$$

This is Max-Cut as a binary quadratic problem.
Unconstrained quadratic 0/1 minimization:

$$\min\ x^T Q x + c^T x \text{ such that } x \in \{0, 1\}^n$$

This is equivalent to Max-Cut, by simple variable transformation.
Q could either be assumed to be upper triangular, or symmetric, with zero diagonal.

Basic semidefinite relaxation

Consider the polytope M_{cut}:=conv$\{xx^T : x \in \{-1, 1\}^n\}$.
It is contained in the set

$$C_n := \{X : diag(X) = e,\ X \succeq 0\}$$

of correlation matrices. Since $x^T Lx = \langle L, xx^T \rangle$ we get

$$z_{mc} = \max\{\langle L, X\rangle : X \in M_{cut}\}$$

$$\leq \max\{\langle L, X\rangle : X \in C_n\} := z_{sdp-basic}$$

Goemans, Williamson (1995) worst-case error analysis (at most 14 % above optimum if weights nonnegative).

Nesterov (1997) error at most 57 % if $L \succeq 0$.

Max-Cut and Cutting Planes

This SDP relaxation can be further tightened by including Combinatorial Cutting Planes:

A simple observation:
Barahona, Mahjoub (1986): Cut Polytope, Deza, Laurent (1997): Hypermetric Inequalities

$$x \in \{-1, 1\}^n, \quad f = (1, 1, 1, 0, \ldots, 0)^T \quad \Rightarrow |f^T x| \geq 1.$$

Results in $x^T f \; f^T x = \langle (xx^T), (ff^T) \rangle = \langle \mathbf{X}, \mathbf{ff^T} \rangle \geq \mathbf{1}$.
Can be applied to any **triangle** $i < j < k$.
Nonzeros of f can also be -1.

There are $4\binom{n}{3}$ such triangle inequality constraints. Direct application of standard methods not possible for $n \approx 100$.

Metric Polytope

We collect all the triangle inequalities

$$x_{ij} + x_{ik} + x_{jk} \geq -1, \; x_{ij} - x_{ik} - x_{jk} \geq -1$$

in the metric polytope MET_n.

Optimizing over MET_n results in a difficult (highly degenerate) LP.

Optimizing over $Met_n \cap C_n$ provides the currently strongest bounds for Max-Cut.

Exact optimization over $MET_n \cap C_n$ feasible if $n \approx 200$.

Can be approximated for large instances, $n \approx 1000$.

Another look at basic relaxation

With $x \in \{-1, 1\}^n$ consider $y \in \{-1, 1\}^n$ with

$$y = x_1 \cdot x = (1, x_1 x_2,\ x_1 x_3,\ \dots, x_1 x_n)^T.$$

Then $yy^T = x_1^2 xx^T = xx^T$.

Anjos, Wolkowicz (2002) and Lasserre (2002) consider

$$y(x) = (1, x_i x_j) \text{ with } 1 \le i < j \le n.$$

and study relaxations based on M_{cut}^2:= conv(yy^T).

Note that M_{cut}^2 is contained in the space of symmetric matrices of order $\binom{n}{2} + 1$.

Basic structural properties of M_{cut}^2

$$
\left(\begin{array}{ccccc|cccccc}
1 & y_{12} & y_{13} & y_{14} & y_{15} & y_{23} & y_{24} & y_{25} & y_{34} & y_{35} & y_{45} \\
 & 1 & y_{23} & y_{24} & y_{25} & y_{13} & y_{14} & y_{15} & z_1 & z_2 & z_3 \\
 & & 1 & y_{34} & y_{35} & y_{12} & z_1 & z_2 & y_{14} & y_{15} & z_4 \\
 & & & 1 & y_{45} & z_1 & y_{12} & z_3 & y_{13} & z_4 & y_{15} \\
 & & & & 1 & z_2 & z_3 & y_{12} & z_4 & y_{13} & y_{14} \\
\hline
 & & & & & 1 & y_{34} & y_{35} & y_{24} & y_{25} & z_5 \\
 & & & & & & 1 & y_{45} & y_{23} & z_5 & y_{25} \\
 & & & & & & & 1 & z_5 & y_{23} & y_{24} \\
 & & & & & & & & 1 & y_{45} & y_{35} \\
 & & & & & & & & & 1 & y_{35} \\
 & & & & & & & & & & 1
\end{array}\right)
$$

Basic Properties (2)

A matrix $Y \in M^2_{cut}$ (of order $\binom{n}{2} + 1$) has the following properties:

• main diagonal equal one, and off diagonal elements of two types:

• y_{ij} for $i < j \leq n$. These are already part of basic $n \times n$ model.

• z_{ijkl} for every 4-tupel of vertices.

These reflect commutativity of the term $x_i x_j x_k x_l$, hence

$$Y_{ij,kl} = Y_{ik,jl} = Y_{il,jk} = z_{ijkl}.$$

$$Q := \{Y : Y \text{ has structure given through } y_{ij}, z_{ijkl},\ Y \succeq 0\}$$

Hierarchy of relaxations

$$z_{SDP} := \max\{\langle L, X\rangle : \; X \in C_n\}$$

$$z_{SDPMET} := \max\{\langle L, X\rangle : X \in C_n \cap MET_n\}$$

$$z_Q := \max\{\langle \tilde{L}, Y\rangle : \; Y \in Q\}$$

Cost function $\tilde{L}$ has Laplacian L as principal submatrix.

$$z_{QMET} := max\{\langle \tilde{L}, Y\rangle : \; Y \in Q \cap MET_{\binom{n}{2}+1}\}$$

$$z_{SDP} \geq z_{SDPMET} \geq z_Q \geq z_{QMET} \geq z_{MC}.$$

and all inequalities can be strict. The only nontrivial condition is $z_{SDPMET} \geq Z_Q$.
The last relaxation z_{QMET} has not been investigated.

A tiny example (n=5) (Laurent (2004))

$$L = -\frac{1}{2}\begin{pmatrix} 0 & 14 & 13 & 14 & 12 \\ 14 & 0 & 13 & 15 & 17 \\ 13 & 13 & 0 & 13 & 11 \\ 14 & 15 & 13 & 0 & 14 \\ 12 & 17 & 11 & 14 & 0 \end{pmatrix}$$

optimize over	optimal value
C_n	38.263
$C_n \cap MET_n$	36.143
Q	34.340
$Q \cap MET_{\binom{n}{2}+1}$	34.000
Max-Cut	34.000

Another example (n=7)

Grishukhin inequality, facet for CUT_7:

optimize over	optimal value
C_n	6.9518
$C_n \cap MET_n$	6.0584
AW	5.7075
Q	5.6152
$Q \cap MET_{\binom{n}{2}+1}$	5.5730
Max-Cut	5.0000

The last three relaxations are intractable for values of n beyond, say $n \approx 30$.

Optimizing over C_n

We solve $\max\langle L, X\rangle : X \in C_n$.
Matrices of order n and $C_n = \{X : diag(X) = e,\ X \succeq 0\}$

n	seconds
1000	12
2000	102
3000	340
4000	782
5000	1570

Computing times to solve the SDP on a my laptop.
Implementation in MATLAB, 30 lines of source code

Practical experience with $C_n \cap MET_n$

graph	n	SDP	SDPMET	time(min)	cut
g1d	100	396.1	352.374	1.10	324
g2d	200	1268.9	1167.978	7.00	1050
g3d	300	2359.6	2215.233	14.01	1953
g1s	100	144.6	130.007	2.60	126
g2s	200	377.3	343.149	8.24	318
g3s	300	678.5	635.039	13.73	555
spin5	125	125.3	109.334	11.40	108

All relaxations solved exactly. The cut value is not known to be optimal.

Optimizing over Q

- Laurent shows that optimizing over Q may provide an improvement by studying facets of M_{cut} for small values of n, $n \leq 7$.
- Solving this relaxation for $n \geq 30$ is nontrivial.

The Q relaxation can be strenthened by adding triangle conditions, leading to optimizing over $Q \cap MET$.

Toward optimizing over $Q \cap MET$

It is computationally prohibitive to work with matrices of order $\binom{n}{2}$.

Our approach: Select small submatrices of Q and impose semidefiniteness as additional constraints (semidefinite cuts).

Starting point is optimal $n \times n$ matrix $Y \in C_n \cap MET_n$.

Main task: Identify small candidate submatrices of Lasserre matrix, which cut off current point Y.

Simplest idea: add K_3

We have $x \in \{-1,1\}^n$ and form $y = y(x) = x_1 \cdot x$. Then $xx^T = yy^T$ and $Y = (y_{ij}) \in C_n \cap MET_n$ is starting point. First Idea: extend $y(x)$ by adding only one term $x_i x_j$ for $2 \leq i < j$.

Hence we consider $y = y(x) := (1, x_1 x_2, \ldots, x_1 x_n, x_i x_j) \in \{-1,1\}^{n+1}$.

Of the $(n+1) \times (n+1)$ matrix yy^T we focus on the submatrix indexed by $(1, [1i], [1j], [ij])$. It contains only elements from Y.

$$Y_{1ij} := \begin{pmatrix} 1 & y_{1i} & y_{1j} & y_{ij} \\ y_{1i} & 1 & y_{ij} & y_{1j} \\ y_{1j} & y_{ij} & 1 & y_{1i} \\ y_{ij} & y_{1j} & y_{1j} & 1 \end{pmatrix}$$

K_3 expansion does not work

Theorem Let $Y \in C_n$. Then $Y \in MET_n$ if and only if $Y_{ijk} \succeq 0 \; \forall i, j, k$
The first direction was already shown by Anjos and Wolkowicz and also Laurent.

If $Y_{ijk} \succeq 0$, then $e^T Y_{1ij} e = 4(1 + x_{ij} + x_{ik} + x_{jk}) \geq 0$. The other inequalities from MET are obtained in a similar way.

If $Y \in MET$, then observe that all principal 3×3 submatrices of Y_{1ij} have same determinant, and the first is by assumption semidefinite. It can be verified that

$$det(Y_{1ij}) = (1 + x_{1i} + x_{1j} + x_{ij})(1 + x_{1i} - x_{1j} - x_{ij})(\ldots)(\ldots),$$

which is ≥ 0 because all factors are nonnegative.
Hence including $Y_{1ij} \succeq 0$ is satisfied once $Y \in C_n \cap MET_n$.

K_3 expansion (2)

Extending $y(x)$ by a single edge $[ij]$ does not improve the relaxation.

If some triangle constraint from MET, say on i, j, k is tight, the proof also shows that the semidefinite matrix Y_{ijk} is singular.

So we try to add edges from a K_4 to extend the relaxation.

Hence we consider $y = y(x) =$ $(1, x_1x_2, x_1x_3, \ldots, x_1x_n, x_ix_j, x_ix_k, x_jx_k) \in \{-1, 1\}^{n+3}$.

The relevant 7×7 submatrix of the Lasserre matrix has all but one entry specified by Y.

The 7×7 submatrix

$$Y_{1ijk}(\xi) := \begin{pmatrix} 1 & y_{1i} & y_{1j} & y_{1k} & y_{ij} & y_{ik} & y_{jk} \\ y_{1i} & 1 & y_{ij} & y_{ik} & y_{1j} & y_{1k} & \xi \\ y_{1j} & y_{ij} & 1 & y_{jk} & y_{1i} & \xi & y_{1k} \\ y_{1k} & y_{ik} & y_{jk} & 1 & \xi & y_{1i} & y_{1j} \\ y_{ij} & y_{1j} & y_{1i} & \xi & 1 & y_{jk} & y_{ik} \\ y_{ik} & y_{1k} & \xi & y_{1i} & y_{jk} & 1 & y_{ij} \\ y_{jk} & \xi & y_{1k} & y_{1j} & y_{ik} & y_{ij} & 1 \end{pmatrix}$$

The unspecified parameter is $\xi = z_{1ijk}$, and represents the products $x_1 x_i x_j x_k$.

K_4 expansion

Note that in case the triangle on i, j, k is tight, the principal submatrix indexed by $[1, ij, ik, jk]$ is singular.

This makes it seem unlikely that Y_{1ikl} is completable to a semidefinite matrix.

If $Y \in C_n$ but $Y \notin MET$, then it is easy to show that Y_{1ikl} may not be sdp completable.

Since we do not only ask for sdp-completability but also that the unspecified parameter ξ is the same throughout, we can not use standard results for sdp-completability.

K_4 extension does not work

Theorem. Let $Y \in C_n \cap MET$ and consider Y_{1ijk} as defined before. Then $Y_{1ijk} \in CUT_7$.

Proof:

The 4×4 submatrix X_4 (of first 4 rows) is part of Y and hence $X_4 \in MET_4 = CUT_4$.
Thus X_4 is a convex combination of cut matrices cc^T of order 4.
If $c = (c_1, c_i, c_j, c_k)$ is such a cut, we consider $\hat{c} := (c_1c_1, c_1c_i, c_1c_j, c_1c_k, c_ic_j, c_ic_k, c_jc_k)$, which has the structure of $y(x)$.
The convex combination with $\hat{c}\hat{c}^T$ instead of the cc^T shows that $Y_{1ijk} \in CUT_7$.

Add CUT_5 constraints

In view of the previous result (and proof), it seems plausible to add constraints of the following type:

Select a 5×5 submatrix of Y, indexed by $i_1 < i_2 < i_3 < i_4 < i_5$ and impose that this submatrix is in CUT_5, hence can be written as a convex combination of $2^4 = 16$ cut matrices C_i.

This is stronger than asking that the associated 11×11 submatrix of the Lasserre matrix is semidefinite.

Preliminary Results

We identify 5-cliques, by looking at all active triangles (at the optimal solution $Y \in C_n \cap MET_n$), and extend them (by enumeration) to 5-cliques, and check whether the resulting submatrix is in CUT_5.
We collect several hundred of these and solve the resulting relaxation, asking that all selected 5-cliques have their submatrix in CUT_5.

n	$C_n \cap MET_n$	with 5-cliques	best known cut
80	1301.7	1294.3	1287
80	1496.6	1487.1	1474
100	1373.9	1358.8	1283
100	130.1	128.8	126

Practicalities are still open

This approach an be combined with triangle separation:

Once a set of 5-cliques is added and the current point is cut off, we may include triangle inequalities until none of them are violated any more.

Then we can again search for violated 5×5 submatrices which are not in CUT_5, and keep iterating.

What are good strategies to identify candidates for 5×5 submatrices?

We use SeDuMi to solve the relaxations, but perhaps some specially taylored code would be more efficient.

Last Slide

The approach can be seen more generally as asking that $Y \in C_n \cap MET_n$ and that all principal $t \times t$ submatrices of Y are in CUT_t. The first nontrivial case is $t = 5$, but $t = 6$ or 7 should also be computationally feasible.

The approach can be extended to other problems, like Stable Set or Coloring.

Computational issues need to be addressed. Is this approach competitive in Branch and Bound schemes?