# Computational Experience with Warmstarting Strategies for Interior Point Methods


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## In this talk

- What is *warmstarting* and when is it applicable?
- Example: Rolling horizon optimization
- Reasons why warmstarting is difficult for interior point methods
- Two new warmstarting schemes
- Example: Robust portfolio selection
- Computational experiments

## Motivation

- Useful for many applications, especially in conic integer optimization
- Lots of phone calls from MOSEK customers

## Warmstarting

$\mathcal{P}$ = an optimization problem (LP, QP, SOCP, QCQP, ...)

$\widehat{\mathcal{P}}$ = a different optimization problem of the same type

- Assume: $x^*$ = solution($\mathcal{P}$) and
- $\mathcal{P} \sim \widehat{\mathcal{P}}$ (different but similar)

| Can we make use of $x^*$ when solving $\widehat{\mathcal{P}}$ ? |
|---|

- Warmstarting:
  - Using $x^*$, compute a "warm" starting point $x^0$
  - initialize algorithm to solve $\widehat{\mathcal{P}}$ starting from $x^0$
- In this talk: we focus on Interior Point Method (=: IPM)

## Why?

Many situations: Need to solve a series of optimization problems:

$$\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \dots$$

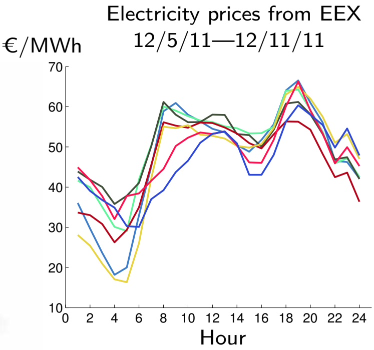
where

$$\mathcal{P}_i \sim \mathcal{P}_{i+1}, \quad i = 1, 2, \dots$$

Examples:

- Rolling horizon optimization (LP, QCQP, ...)
- Efficient frontier computation (QP, SOCP)
- Relaxations in integer programming (LP, QP, SDP)
- If you just have a confident solution guess

## Example: Rolling horizon optimization of charging of PEVs

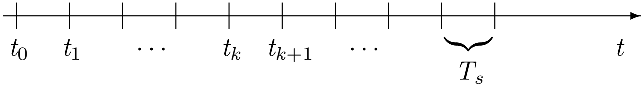



Given driving schedule, when to charge to minimize cost?

## Simple (discrete time) battery model

$$x_{k+1} = x_k + T_s(\eta/Q_n)u_k - T_s d_k$$

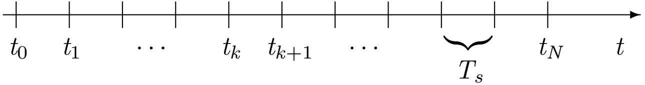
$$
\begin{aligned}
x_k &= \text{battery power storage at time } t_k,\ x_k \in [0,1] \\
u_k &= \text{charging power at time } t_k \\
d_k &= \text{driving at time } t_k \\
T_s &= \text{time interval length} \\
\eta &= \text{charger efficiency} \\
Q_n &= \text{nominal capacity of battery}
\end{aligned}
$$



## Economic Model Predictive Control

$$
(\text{LP}) \begin{cases} \min\limits_{x,u} & \displaystyle\sum_{i=0}^{N-1} p_k u_k \\ \text{s.t.} & x_{k+1} = x_k + T_s(\eta/Q_n)u_k - T_s d_k & k \in \mathcal{N} \\ & u_{\mathsf{min}} \le u_k \le u_{\mathsf{max},k} & k \in \mathcal{N} \\ & 0.2 \le x_k \le 0.8 & k \in \mathcal{N} \end{cases}
$$

$$
\begin{aligned}
\mathcal{N} &= \{0, 1, 2, \ldots, N-1\} \\
x_k &= \text{battery power storage at time } t_k \\
u_k &= \text{charging power at time } t_k \\
p_k &= \text{(predicted) electricity price at time } t_k \\
d_k &= \textit{given} \text{ driving schedule} \\
u_{\mathsf{max},k} &= P_{\mathsf{max}}, \text{ but } 0 \text{ if } d_k > 0
\end{aligned}
$$

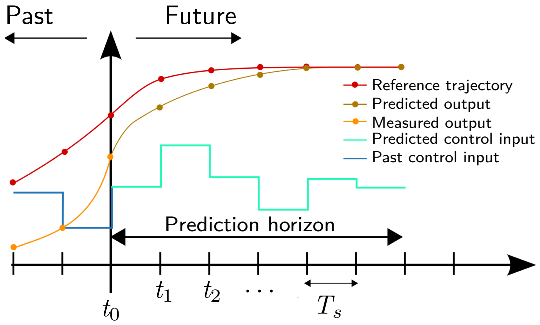


**Model Predictive Control Loop:**

**for** $k = 0, 1, 2, \dots,$ **do**

- Solve (LP) and obtain $\boldsymbol{u}_k^* = (u_k^{(0)}, u_k^{(1)}, \dots, u_k^{(N-1)})$
- Apply $u_k^{(0)}$ at time $t_k$ to system

**end**

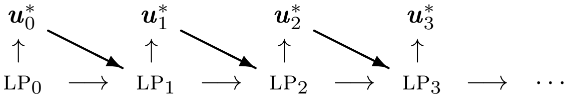


- This is series of related(!) linear programs
- Good reason to believe $\boldsymbol{u}_{k+1}^*$ "similar" to $\boldsymbol{u}_k^*$
- Therefore: We should utilize information from solution of problem $k$ when solving problem $k+1$

$$
\begin{array}{ccccccccc}
\boldsymbol{u}_0^* & & \boldsymbol{u}_1^* & & \boldsymbol{u}_2^* & & \boldsymbol{u}_3^* & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathrm{LP}_0 & \longrightarrow & \mathrm{LP}_1 & \longrightarrow & \mathrm{LP}_2 & \longrightarrow & \mathrm{LP}_3 & \longrightarrow & \cdots
\end{array}
$$

- This is series of related(!) linear programs
- Good reason to believe $\boldsymbol{u}_{k+1}^*$ "similar" to $\boldsymbol{u}_k^*$
- Therefore: We should utilize information from solution of problem $k$ when solving problem $k+1$

- Warmstarting:



## Problem Perturbation I

$$\mathcal{P} = \left\{ \min_x c^T x, \ \ \text{s.t. } Ax \leq b \right\}$$

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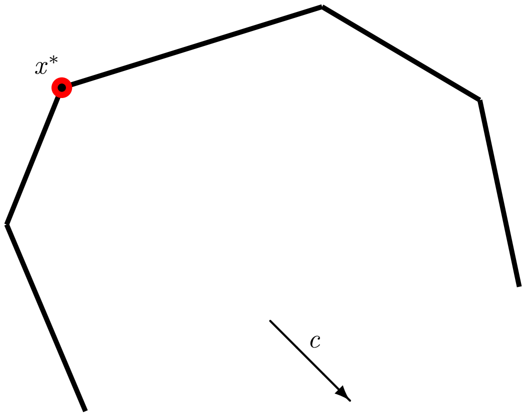

## Problem Perturbation II

$$\mathcal{P} = \left\{ \min_x c^T x, \ \ \text{s.t. } Ax \leq b \right\}$$

$$\widehat{\mathcal{P}} = \left\{ \min_x c^T x, \ \ \text{s.t. } \widehat{A}x \leq \widehat{b} \right\}$$

## Problem Perturbation II

$$\mathcal{P} = \left\{ \min_x c^T x, \ \ \text{s.t. } Ax \le b \right\}$$

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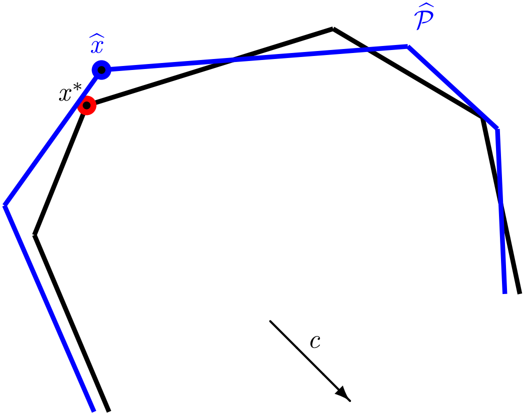
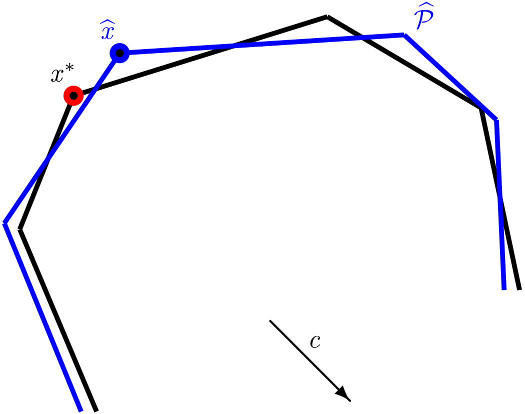


## Problem Perturbation III

$$\mathcal{P} = \left\{ \min_x c^T x, \ \ \text{s.t. } Ax \le b \right\}$$

$$\widehat{\mathcal{P}} = \left\{ \min_x c^T x, \ \ \text{s.t. } \widehat{A}x \le \widehat{b} \right\}$$



## Problem Perturbation IV

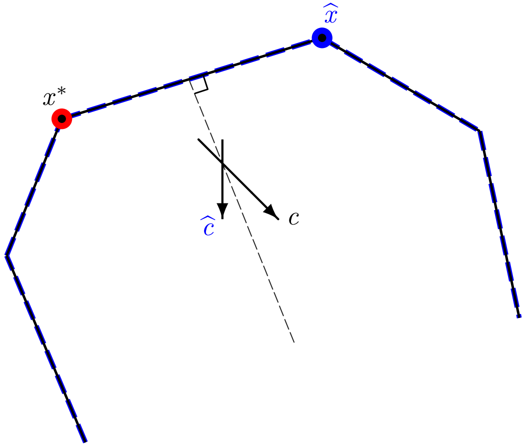
$$\mathcal{P} = \left\{ \min_x c^T x, \;\; \text{s.t. } Ax \le b \right\}$$

$$\widehat{\mathcal{P}} = \left\{ \min_x \widehat{c}^T x, \;\; \text{s.t. } Ax \le b \right\}$$

## Problem Perturbation IV

$$\mathcal{P} = \left\{ \min_x c^T x, \;\; \text{s.t. } Ax \leq b \right\}$$

$$\widehat{\mathcal{P}} = \left\{ \min_x \widehat{c}^T x, \;\; \text{s.t. } Ax \leq b \right\}$$



## Many Problems Can Happen

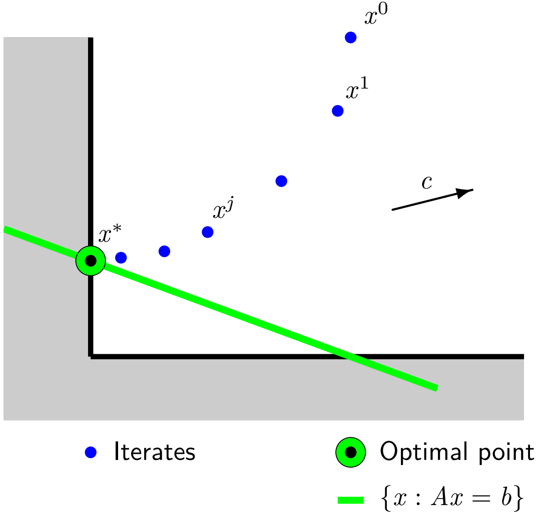
These examples show

- $x^*$ might be infeasible in $\widehat{\mathcal{P}}$
- although $\mathcal{P} \approx \widehat{\mathcal{P}}$, solution may "jump"
- problem may even change status: e.g. from feasible to infeasible
- $x^*$ might be on or close to boundary in $\widehat{\mathcal{P}}$ (algorithmic problem)

| solution $x^*$ is not a continuous function of the data $(A, b, c)$. |
| --- |

## Interior Point Algorithms for LP

$\textsc{LP}(A, b, c) = \left\{\min_x c^T x, \ \text{s.t. } Ax = b, \ x \geq 0\right\}$



## Warmstarting Research

- The Simplex Method: works well.
- Active set: Often works well! (though no guarantee).
- IPMs are perceived fundamentally deficient w.r.t. warmstarting
  - $x^*$ on boundary of feasible region for $\mathcal{P}$
  - close to the boundary, IPMs behave badly
- Previously tried for IPMs:
  - Solve $\mathcal{P}$ with IPM, store *all* iterates:

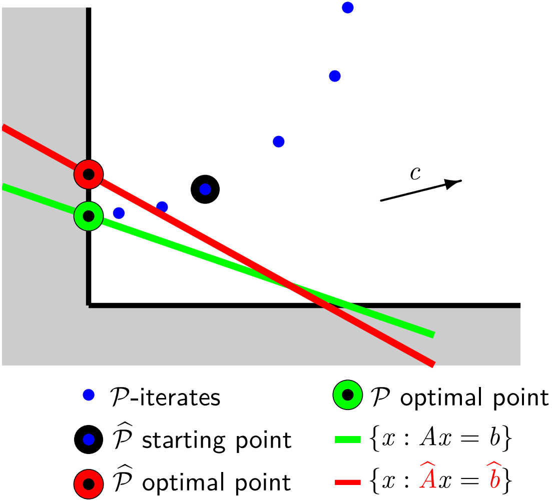
$$I = \left\{ x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(\text{final})} \approx x^\star \right\}$$

  - search $I$ for an element that "looks good" for $\widehat{\mathcal{P}}$.

# Warmstarting Research for IPM

Original LP: $\mathcal{P} = \text{LP}(A, b, c) = \left\{\min_x c^T x, \text{ s.t. } Ax = b, \ x \geq 0\right\}$

Perturbed LP: $\widehat{\mathcal{P}} = \text{LP}(\widehat{A}, \widehat{b}, c) = \left\{\min_x c^T x, \text{ s.t. } \widehat{A}x = \widehat{b}, \ x \geq 0\right\}$



**A practical problem with the approach**

- Optimization algorithms are used as black-box routines
- Usually no output of intermediate iterates
- Only output is primal solution and sometimes also dual.

**Our goal:**

Warmstarting procedure using only

- primal optimal or final solution of $\mathcal{P}$ or
- primal and dual optimal or final solution of $\mathcal{P}$

when initializing algorithm to solve $\widehat{\mathcal{P}}$.

## Homogeneous and Self-Dual Model for Linear Programming

Given a linear program $\text{LP}(A, b, c) = \left\{\min_x c^T x, \ \text{ s.t. } Ax = b, \ x \geq 0\right\}$, find $(x, \tau, y, s, \kappa)$ that satisfies

$$
\begin{aligned}
Ax - b\tau &= 0 \\
-A^T y - s + c\tau &= 0 \\
-c^T x + b^T y - \kappa &= 0 \\
(x, \tau) \geq 0, \ (s, \kappa) \geq 0, \ y &\in \mathbb{R}^m
\end{aligned}
$$

- If $\tau > 0$ then $(x, y, s)/\tau$ is optimal for $\text{LP}(A, b, c)$.
- If $\kappa > 0$ then $\text{LP}(A, b, c)$ is infeasible.

The convergence efficiency is measured by the primal-dual potential function:

$$\phi(x, s) = (n + \rho) \log(x^T s) - \sum_{j=1}^{n} \log(x_j s_j).$$

## Initialization of algorithm to solve HSD-model

- Usually, $(x^0, \tau^0, y^0, s^0, \kappa^0) = (e, 1, 0, e, 1)$ is used (=: cold-start) where $e := (1, 1, \ldots, 1)$ and $\phi(x^0, s^0) = \rho \log(n)$.

## Our warmstarting schemes

- When only primal solution $x^*$ is available:

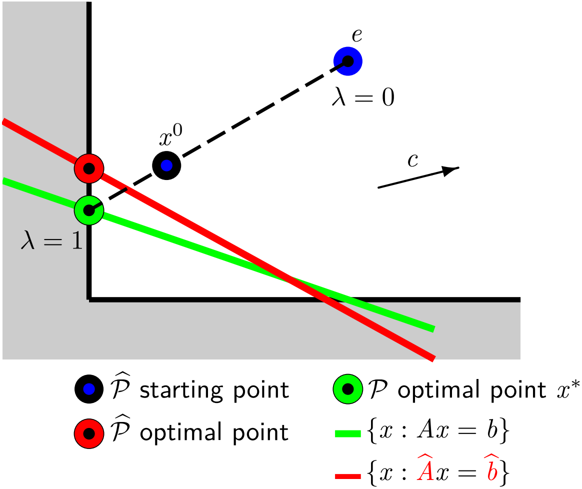
$$
(\mathrm{W_P}) \left\{ \begin{array}{rcl} x^0 & = & \lambda x^* + (1-\lambda) e \\ s^0 & = & \mu^0 (x^0)^{-1} \\ y^0 & = & 0 \\ \tau^0 & = & 1 \\ \kappa^0 & = & \mu^0 \end{array} \right.
$$

where

- $\lambda \in [0, 1]$, $\mu^0 > 0$
- $(x^0)^{-1}$ denotes the *elementwise* reciprocal of $x^0$

$x^0 = \lambda x^* + (1-\lambda)e$

$s^0 = \mu^0 (x^0)^{-1}$




$(1-\lambda)e$ added to $x^0$ to ensure interiority (needed for IPM)

$s^0$ chosen so that $x^0 \circ s^0 = \mu^0 e$, where $\circ :=$ elementwise product.

## Our warmstarting schemes

- When both primal $x^*$ and dual solutions $(y^*, s^*)$ are available:

$$(\mathrm{W}_{\mathrm{PD}}) \left\{ \begin{array}{rcl} x^0 & = & \lambda x^* + (1-\lambda)e \\ s^0 & = & \lambda s^* + (1-\lambda)e \\ y^0 & = & \lambda y^* \\ \tau^0 & = & 1 \\ \kappa^0 & = & (x^0)^T s^0 / n \end{array} \right.$$

- Also $y^0 = \lambda y^* + (1-\lambda)0$.

If new primal variables and/or new dual variables are added, they are set to default values without warmstarting.

## Our warmstarting schemes

$$(\mathrm{W_P}) \begin{cases} x^0 &= \lambda x^* + (1-\lambda)e \\ s^0 &= \mu^0 (x^0)^{-1} \\ y^0 &= 0 \\ \tau^0 &= 1 \\ \kappa^0 &= \mu^0 \end{cases} \qquad (\mathrm{W_{PD}}) \begin{cases} x^0 &= \lambda x^* + (1-\lambda)e \\ s^0 &= \lambda s^* + (1-\lambda)e \\ y^0 &= \lambda y^* \\ \tau^0 &= 1 \\ \kappa^0 &= (x^0)^T s^0 / n \end{cases}$$

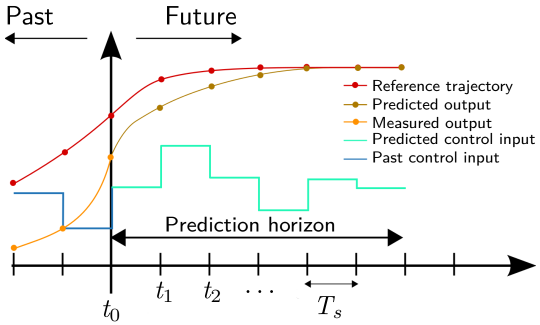
- $\mathrm{W_P}$ suited when
  - Just $x^*$ is available (black box)
  - Just one problem is to be solved, but you have a "good guess"
- $\mathrm{W_{PD}}$ suited when
  - $(x^*, y^*, s^*)$ is available (better black box) but still no intermediate iterates

## Warmstarting for the electric vehicle example

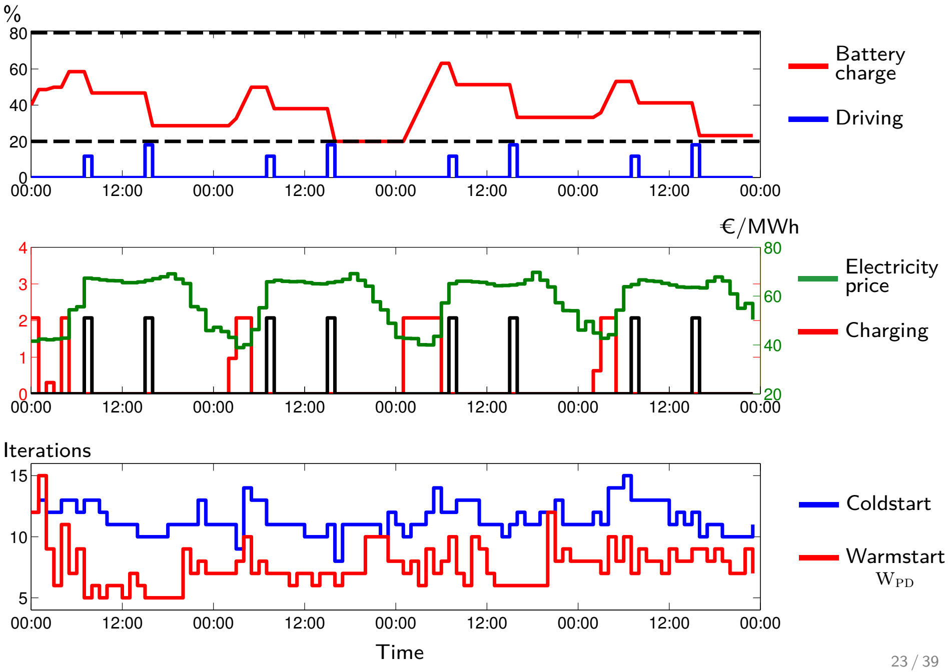
- $\boldsymbol{u}_k^* = (u_k^{(0)}, u_k^{(1)}, \ldots, u_k^{(N-1)})$ solution at time $t_k$
- Then in place of "$x^*$" in warmstarting schemes, we use

$$(u_k^{(1)}, \ldots, u_k^{(N-1)}, u_k^{(N-1)})$$

i.e. $\boldsymbol{u}_k^*$ translated one place



# Charging Schedule for Electric Vehicle and Warmstarting Performance



## Linear Programs from NETLIB

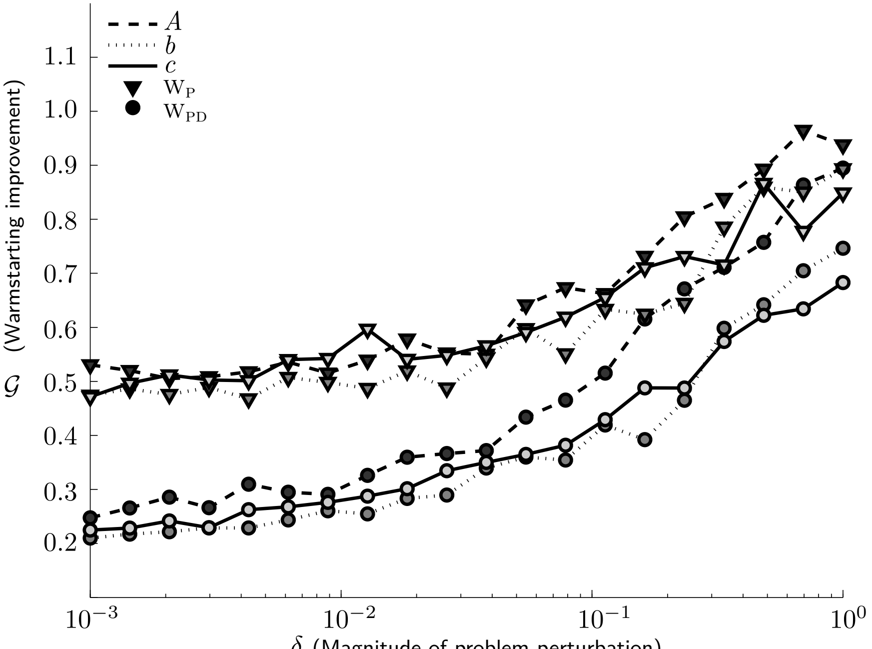
- $\sim$90 real-life Linear Programs, varying size and sparsity
- For all problems, do
  - Solve $\mathcal{P}$. Optimal solution: $(x^*, y^*, s^*)$
  - Generate $\widehat{\mathcal{P}}$ by a *random perturbation* of $\mathcal{P}$:
    - $\widehat{A} = A + \delta \Delta A$ or $\widehat{b} = b + \delta \Delta b$ or $\widehat{c} = c + \delta \Delta c$
    - $\delta$ measures perturbation magnitude
  - Solve $\widehat{\mathcal{P}}$ coldstarting and warmstarting using $x^*$ and $(y^*, s^*)$
- Measure of warmstarting improvement:

$$\mathcal{R} = \frac{\#\text{Iterations to solve } \widehat{\mathcal{P}} \text{ warmstarted}}{\#\text{Iterations to solve } \widehat{\mathcal{P}} \text{ coldstarted}}$$

- and entire problem set:

$$\mathcal{G} = \sqrt[K]{\mathcal{R}_1 \cdots \mathcal{R}_K}$$

## Warmstarting Performance on NETLIB Linear Programs



**Theoretical Justification**

- The primal-dual potential function initial value remains bounded by $O(\rho \log(n))$ for any fixed $0 \leq \lambda < 1$.

- Conservative approach requires $\lambda \ll 1$
- In practice: use much more aggressive choice of $\lambda$ (i.e. close to $1$)
- For experiments above: $\lambda = 0.99$.
- Similar results for $\mathrm{w_P}$

Anders Skajaa, Erling D. Andersen and Yinyu Ye. *Warmstarting the Homogeneous and Self-Dual Interior Point Method for Linear and Conic Quadratic Problems.* Working paper to appear in Math. Prog. Computation.

## Portfolio selection and efficient frontier

- Available for investment: $n$ different assets
- Denote

$$\begin{aligned} r_i &= \text{random variable, return of asset } i \\ r &= \text{vector stacking the } r_i \end{aligned}$$

- Assume

$$r \sim \mathcal{N}(\mu, \Sigma)$$

where

$$\begin{aligned} \mu &= \text{mean returns} \\ \Sigma &= \text{covariance matrix} \end{aligned}$$

## Classical Markowitz portfolio selection

$r_i$ = RV, return of asset $i$
$r$ = vector stacking the $r_i$

$$r \sim \mathcal{N}(\mu, \Sigma)$$

- $\phi_i$ = fraction of wealth in asset $i$
- $\phi$ = vector stacking the $\phi_i$ (entire portfolio)
- Then

  *Expected* return of portfolio $\phi$ is

$$E[r^T\phi] = \mu^T\phi$$

  "risk" of portfolio $\widehat{=}$

$$\mathrm{Var}(r^T\phi) = \phi^T\Sigma\phi$$

$\mu^T\phi$ = expected return of $\phi$
$\phi^T\Sigma\phi \hat{=}$ risk of $\phi$

## Classical Markowitz portfolio selection

- Markowitz portfolio optimization:

  Optimize a trade-off between max(return) and min(risk)

- Assuming we know **with certainty** the data $(\mu, \Sigma)$,

  we can compute the classical Markowitz portfolio from:

$$(\text{QP}) \begin{cases} \text{minimize}_\phi & \phi^T\Sigma\phi \\ \text{subject to} & \mu^T\phi \geq t \\ & e^T\phi = 1 \\ & \phi \geq 0 \end{cases}$$

i.e.: minimize variance s.t. expected return $\geq t$

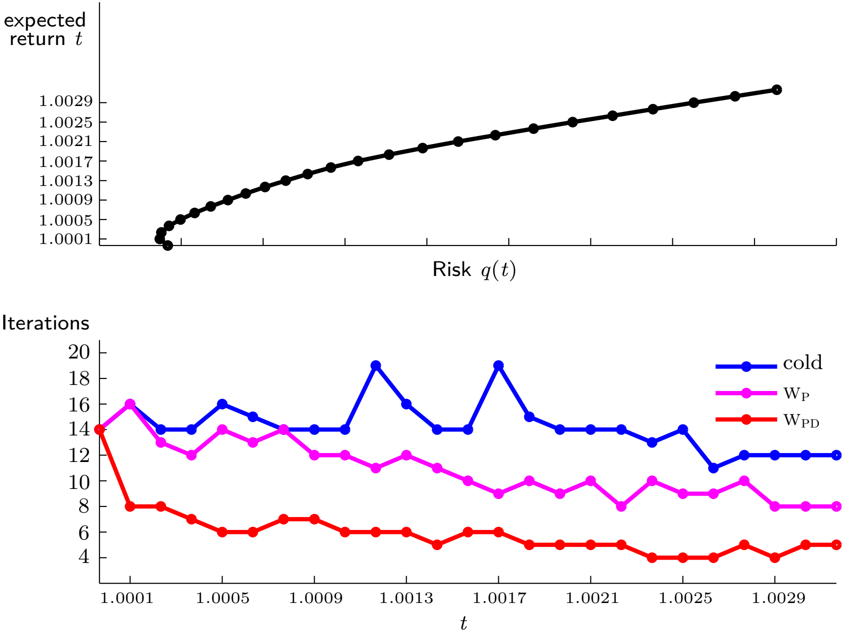
$$(\text{QP}) \begin{cases} \text{minimize}_\phi & \phi^T \Sigma \phi \\ \text{subject to} & \mu^T \phi \geq t \\ & e^T \phi = 1 \\ & \phi \geq 0 \end{cases}$$

## Efficient frontier

$t =$ demanded minimal expected return

- Denote the minimum risk by $q(t)$
- Efficient frontier: $(t, q(t))$ for a range of $t$
- A series of related QPs, use warmstarting!
- Data:
  - $500$ assets from S&P 500 index
  - expected returns $\mu$ and covariances $\Sigma$ estimated from $800$ daily closing prices 2007–2011

## Warmstarting performance when computing the efficient frontier



$r_i =$ RV, return of asset $i$
$r\ =$ vector stacking the $r_i$

## *Robust* portfolio selection

- Now assume

$$r \sim \mathcal{N}(\mu, \Sigma) \quad \text{where} \quad \Sigma = V^T F V + D$$

- and data in **uncertainty** sets:

$$\begin{aligned} \mu \in S_\mu &:= \{\mu : \mu = \mu_0 + \xi,\ |\xi_i| \le \gamma_i\} \\ D \in S_D &:= \left\{D : D = \mathsf{diag}(d),\ 0 \le d_i \le \bar{d}_i\right\} \\ V \in S_V &:= \{V : V = V_0 + W,\ \|W_{:i}\|_G \le \bar{w}_i\} \end{aligned}$$

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D. Goldfarb and G. Iyengar. *Robust portfolio selection problems.* Math. Oper. Res., Feb. 2003.

## ***Robust*** **portfolio selection**

$r_i =$ RV, return of asset $i$
$r \;=$ vector stacking the $r_i$
$r \;\sim \mathcal{N}(\mu, \Sigma), \quad \Sigma = V^T F V + D$
$\mu^T \phi \quad =$ expected return of $\phi$
$\phi^T \Sigma \phi \hat{=}$ risk of $\phi$
$S_x =$ uncertainty set of $x$

- Find portfolio $\phi$ minimizing *worst-case* risk (variance):

Robust portfolio selection

$$\begin{cases} \text{minimize}_\phi & \max_{V \in S_V, D \in S_D} \{\phi^T \Sigma \phi\} \\ \text{subject to} & \min_{\mu \in S_\mu} \{\mu^T \phi\} \geq t \\ & e^T \phi = 1, \quad \phi \geq 0 \end{cases}$$

$r_i =$ RV, return of asset $i$
$r$ = vector stacking the $r_i$
$r \sim \mathcal{N}(\mu, \Sigma), \quad \Sigma = V^T F V + D$
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## ***Robust*** **portfolio selection**

- Find portfolio $\phi$ minimizing *worst-case* risk (variance):

Robust portfolio selection

$$\begin{cases} \text{minimize}_\phi & \max_{V \in S_V, D \in S_D} \{\phi^T \Sigma \phi\} \\ \text{subject to} & \min_{\mu \in S_\mu} \{\mu^T \phi\} \geq t \\ & e^T \phi = 1, \quad \phi \geq 0 \end{cases}$$

Compare with classical

$$\begin{cases} \text{minimize}_\phi & \phi^T \Sigma \phi \\ \text{subject to} & \mu^T \phi \geq t \\ & e^T \phi = 1, \ \phi \geq 0 \end{cases}$$

## ***Robust*** **portfolio selection**

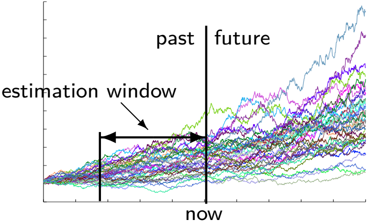
- The robust portfolio selection problem $\left\{ \begin{array}{rl} \min_\phi & \max_{V \in S_V, D \in S_D} \{\phi^T \Sigma \phi\} \\ \text{s.t.} & \min_{\mu \in S_\mu} \{\mu^T \phi\} \geq t \\ & e^T \phi = 1, \quad \phi \geq 0 \end{array} \right.$

  can be formulated as equivalent **S**econd **O**rder **C**one **P**rogram:

$$
\begin{array}{ll}
\min_{\{\phi,\dots\}} & \nu + \delta \\
\text{subject to} & \mu_0^T \phi - \gamma^T \psi \geq t \\
& r \geq w^T \psi, \;\; -\phi \leq \psi \leq \phi \\
& e^T \phi = 1, \;\; \phi \geq 0 \\
& \tau + e^T t \leq \nu, \;\; \sigma \leq 1/\lambda_{\mathsf{max}}(H) \\
& \|(2r, \sigma - \tau)\|_2 \leq \sigma + \tau \\
& \|(2v_i, 1 - \sigma\lambda_i - t_i)\|_2 \leq 1 - \sigma\lambda_i + t_i, \;\; i = 1, \dots, m \\
& \|(2\bar{D}^{1/2}\phi, 1 - \delta)\|_2 \leq 1 + \delta
\end{array}
$$

- SOCPs can be solved as efficiently as QPs
- Warm points generalized via Jordan algebra operations associated with convex cones

## ***Robust*** **portfolio selection**




- Frequent *rebalancing* of portfolio:

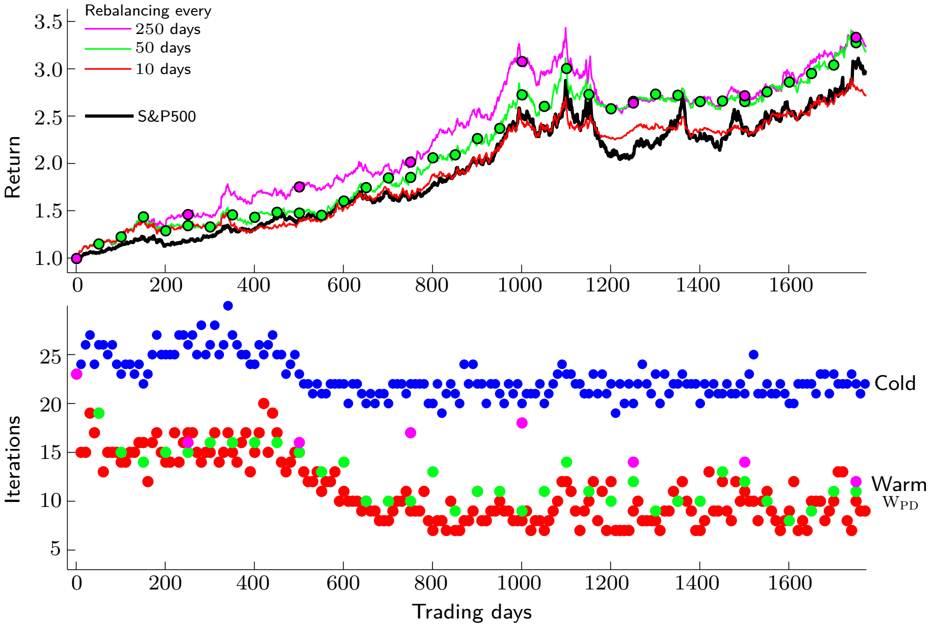
  **repeat**:

  - Estimate problem-data $\mu_0, \gamma, \ldots$ and uncertainty sets based on observed data from previous time window
  - Rebalance portfolio by solving robust portfolio selection problem

  **end**

- Series of related SOCPs
- Faster solution $\longrightarrow$ more frequent rebalancing
- Warmstarting! (use previous portfolio as "$x^*$").

# Portfolio rebalancing and warmstarting performance



## Concluding remarks

- Warmstarting schemes seem effective in practice
  - Easy to compute
  - Require *only* final solution of $\mathcal{P}$ (OK with black-boxes)
  - Significant work reductions in practice
  - Work at least for LP, QP, SOCP
- More details and examples in working paper:
  - More linear programs and rolling horizon conic examples
  - QP subproblems in cutting-plane/bundle methods
- Future
  - Easily extend-able to SDP, but computational experiments remain to be seen
  - Applications in integer optimization with branching and cutting
  - Theoretical question: if the perturbation is random, prove that the scheme works with high probability!

## Concluding remarks

- Warmstarting schemes seem effective in practice
  - Easy to compute
  - Require *only* final solution of $\mathcal{P}$ (OK with black-boxes)
  - Significant work reductions in practice
  - Work at least for LP, QP, SOCP
- More details and examples in working paper:
  - More linear programs and rolling horizon conic examples
  - QP subproblems in cutting-plane/bundle methods
- Future
  - Easily extend-able to SDP, but computational experiments remain to be seen
  - Applications in integer optimization with branching and cutting
  - Theoretical question: if the perturbation is random, prove that the scheme works with high probability!

**Thank you!**

## Linear Programs from NETLIB

- We define $\widehat{\mathcal{P}}$ by *randomly* perturbing data $A, b$ and $c$:
  - Assume $v \in \mathbb{R}^M$ is vector we want to perturb
  - $s =$ random number, $[0, 1]$-uniform
  - if $s \leq \min\{0.1, 20/M\}$

$$v_i := \begin{cases} \delta r & \text{if } |v_i| \leq 10^{-6} \\ (1 + \delta r) v_i & \text{otherwise} \end{cases}$$

    where $r =$ random number, $[-1, 1]$-uniform
  - otherwise don't change
- On average, $\min\{10\%, 20\}$ of the elements are changed
- "Magnitude" of perturbation measured by $\delta$

## Example: Portfolio selection and efficient frontier

- Available for investment: $n$ different assets
- Denote

$$\begin{aligned} r_i &= \text{random variable, return of asset } i \\ r &= \text{vector stacking the } r_i \end{aligned}$$

- Assume

$$r = \mu + V^T f + \epsilon$$

  where

$$\begin{aligned} \mu &= \text{mean returns} \\ f &= \text{random returns of ``factors'' that drive market} \\ V &= \text{factor loading matrix, } V \in \mathbb{R}^{m \times n} \\ \epsilon &= \text{``residuals'' assumed normally distributed} \end{aligned}$$

## Classical Markowitz portfolio selection

$$\begin{aligned} r_i &= \text{RV, return of asset } i \\ r &= \text{vector stacking the } r_i \\ r &= \mu + V^T f + \epsilon \end{aligned}$$

- Assume

$$\epsilon \sim \mathcal{N}(0, D)$$
$$f \sim \mathcal{N}(0, F)$$

- Then

$$r \sim \mathcal{N}(\mu, \Sigma) \quad \text{where} \quad \Sigma = V^T F V + D$$

- $\phi_i =$ fraction of wealth in asset $i$ and $\phi =$ entire portfolio
- Then

*Expected* return of portfolio $\phi$ is

$$E[r^T \phi] = \mu^T \phi$$

"risk" of portfolio $\widehat{=}$

$$\text{Var}(r^T \phi) = \phi^T \Sigma \phi$$