Distributions of Demmel and Related Condition Numbers: Finite and Asymptotic Behaviors

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[Joint work with Matthew McKay and Yang Chen] Demmel introduced a scaled condition number in an attempt to characterize the degree of difficulty associated with a numerical analysis problem. In particular, he used this so-called Demmel condition number to model the probability that a matrix inversion problem is difficult. The Demmel and other related condition numbers have found numerous applications in various branches of science and engineering including multivariate analysis and wireless communications.

Consider a random matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$ $(m \geq n)$ containing independent complex Gaussian entries with zero mean and unit variance, and let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n < \infty$ denote the eigenvalues of $\mathbf{A}^*\mathbf{A}$ where $(\cdot)^*$ represents conjugate-transpose. This talk presents some new results on the distribution of the random variables $\frac{\sum_{j=1}^{n}\lambda_j}{\lambda_k}$, for $k = 1$ and $k = 2$. These two variables are related to certain condition number metrics, including the so-called Demmel condition number. For both cases, we derive new exact expressions for the probability densities, and establish the asymptotic behavior as the matrix dimensions grow large. In particular, it is shown that as n and m tend to infinity with their difference fixed, both densities scale on the order of n^3. After suitable transformations, we establish exact expressions for the asymptotic densities, obtaining simple closed-form expressions in some cases. Our results generalize the work of Edelman on the Demmel condition number for the case $m = n$.