Exactly solvable random polymers and their continuum scaling limits

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Three random polymer models

Discrete polymer with log-gamma weights.

Seppäläinen (2009), Georgiou-Seppäläinen (2011), Corwin-O'C-Seppäläinen-Zygouras (2011).

Semi-discrete polymer.

O'C-Yor (2002), O'C-Moriarty (2007), Seppäläinen-Valko (2010), O'C (2009), Borodin-Corwin (2011), Borodin-Corwin-Ferrari (2012).

Continuum random polymer.

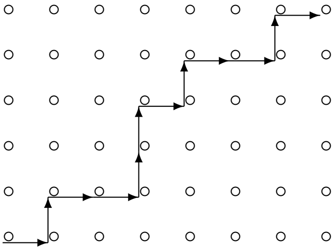
Amir, Corwin, Quastel (2010), Sasamoto, Spohn (2010), Calabrese, Le Doussal, Rosso (2010), Dotsenko (2010).

Discrete polymer

Weights w_{ij}, iid distributed as $-\log\Gamma(\theta)$. Partition function

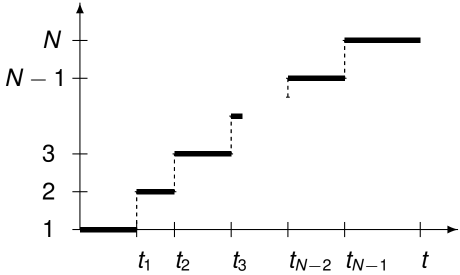
$$Z_{m,n} = \sum_{\phi \in \Pi_{m,n}} e^{\sum_{(i,j)\in\phi} w_{ij}}.$$

A lattice path in $\Pi^1_{8,6}$:



Semi-discrete polymer

A path $\phi \equiv \{0 < t_1 < \ldots < t_{N-1} < t\}$:

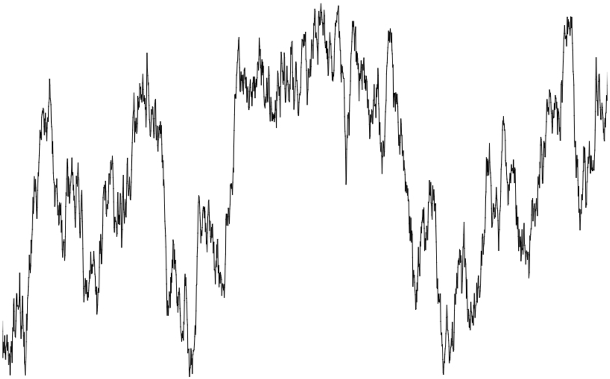


$B_1, B_2, \ldots$ independent Brownian motions.

$$E(\phi) = B_1(t_1) + B_2(t_2) - B_2(t_1) + \cdots + B_N(t) - B_N(t_{N-1}).$$

$$Z_t^N(\beta) = \int e^{\beta E(\phi)} d\phi.$$

A Brownian bridge



Continuum random polymer

$$Z(t,x,y) = \mathbb{E}_x \left[:\exp: \left(\int_0^t \xi(s,\beta_s)ds \right) \right]$$

where β is a Brownian bridge from x to y in time t and ξ is space-time white noise. This is defined by the chaos expansion

$$\begin{aligned} Z(t,x,y) = \\ & 1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k((t_1,x_1),(t_2,x_2),\dots,(t_k,x_k)) \\ & \qquad \times \xi(t_1,x_1)\xi(t_2,x_2)\cdots\xi(t_k,x_k)dt_1 dx_1 \cdots dt_k dx_k, \end{aligned}$$

where $\Delta_k(t) = \{0 < t_1 < \cdots < t_k < t\}$ and R_k are the k-point functions for the Brownian bridge.

Related SPDEs

Let $p(t,x,y) = (2\pi t)^{-1/2} e^{(x-y)^2/2t}$. The function $u = pZ$ satisfies the stochastic heat equation

$$\partial_t u = \frac{1}{2}\partial_y^2 u + \xi(t,y)u$$

with $u(0,x,y) = \delta(x-y)$, and $h = \log u$ is the Cole-Hopf solution of the KPZ equation

$$\partial_t h = \frac{1}{2}\partial_y^2 h - \frac{1}{2}(\partial_y h)^2 + \xi(t,y),$$

with ‘narrow wedge’ initial condition.

Interpretations of continuum model

The continuum random polymer is a scaling limit of discrete directed polymer models, in the ‘intermediate disorder’ regime (Alberts-Khanin-Quastel 2010/12).

Moreover, $h = \log u$ arises as the scaling limit of the height profile of the weakly asymmetric simple exclusion process (Bertini-Giacomin 1997). With this ‘surface growth’ interpretation, h is understood to be the physically relevant solution to the KPZ equation.

The law of Z

Recent development: exact distribution of $Z(t,x,y)$. Generating function given as a Fredholm determinant.

Two approaches.

1. Uses ASEP approximation together with recent work by Tracy and Widom on exact formulas for ASEP. Amir-Corwin-Quastel (2010), Sasamoto-Spohn (2010).

2. Replicas: moments of the partition function are related to the attractive δ-Bose gas. Calabrese, Le Doussal, Rosso (2010), Dotsenko (2010).

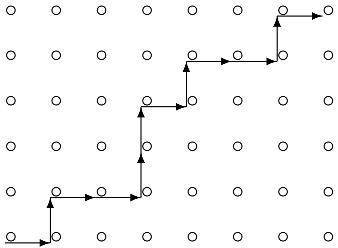
Corollary: As $t\to\infty$, distribution of $\log Z(t,x,y)$ rescales to the Tracy-Widom F_2 distribution.

Discrete polymer

Partition function

$$Z_{m,n}(\beta) = \sum_{\phi \in \Pi_{m,n}} e^{\beta \sum_{(i,j)\in\phi} w_{ij}}.$$

A lattice path in $\Pi^1_{8,6}$:



Last passage percolation and random matrices

Zero-temperature limit:

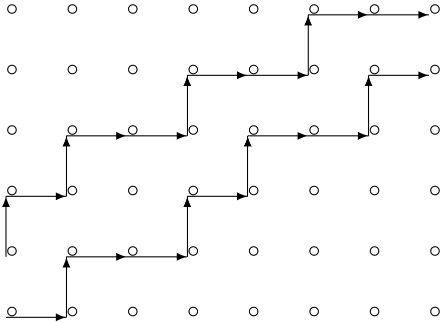
$$\lim_{\beta\to\infty}\frac{1}{\beta}\log Z_{m,n}(\beta)=\max_{\phi\in\Pi_{m,n}}\sum_{(i,j)\in\phi}w_{ij}=:G_{m,n}$$

Theorem (Johansson 2000)

If weights are iid standard exponential random variables then $G_{m,n}$ has the same law as the largest eigenvalue of $A^\dagger A$, where A is a $m\times n$ matrix with standard complex Gaussian entries.

Proof is based on the RSK (Robinson-Schensted-Knuth) correspondence and extends to other eigenvalues.

Non-intersecting lattice paths



A pair of non-intersecting lattice paths in $\Pi^2_{8,6}$

RSK and random matrices

Let w_{ij} be independent standard exponential random variables. Define $L_1 \geq L_2 \geq \cdots \geq L_n$ by

$$L_1 + \cdots + L_k = \max_{\Gamma \in \Pi^k_{m,n}} \sum_{(i,j)\in\Gamma} w_{ij}.$$

Theorem (Various results combined)

L has the same distribution as the eigenvalues of $A^\dagger A$.

This distribution is called the Laguerre ensemble. The density is proportional to

$$\prod_{i<j\leq n} (x_i - x_j)^2 \prod_{i\leq n} x_i^{m-n} e^{-x_i} dx.$$

'Tropical RSK correspondence'

A.N. Kirillov (2000) introduced a 'tropical' analogue of the RSK correspondence, defined by replacing $(\max, +)$ by $(+, \times)$.

In this setting we define

$$Y_1 \cdots Y_k = \sum_{\Gamma \in \Pi^k_{m,n}} \prod_{(i,j) \in \Gamma} d_{ij}.$$

Note that setting $d_{ij} = e^{w_{ij}}$ gives $Y_1 = Z_{m,n}$.

Tropical RSK with random input

Theorem (Corwin-Sepäläinen-O'C-Zygouras 2011)

Let d_{ij} be independent inverse gamma random variables with respective parameters $\hat{\theta}_i + \theta_j$. Then the distribution $\mu^{n,m}_{\theta,\hat{\theta}}$ of $Y = (Y_1, \ldots, Y_n)$ is given in terms of $GL(n,\mathbb{R})$-Whittaker functions. For example,

$$\mu^{n,n}_{\theta,\hat{\theta}}(dy) = \prod_{i,j} \Gamma(\theta_i + \hat{\theta}_j)^{-1} e^{-y_n^{-1}} \Psi_\theta(y) \Psi_{\hat{\theta}}(y) \prod_i \frac{dy_i}{y_i}.$$

The law of the partition function $Y_1 = Z_{m,n}$

Corollary

For $s > 0$,

$$Ee^{-sZ_{m,n}} = \int s^{\sum(\theta_i - \lambda_i)} \prod_{i,j} \Gamma(\lambda_i - \theta_j) \prod_{i,j} \frac{\Gamma(\lambda_i + \hat{\theta}_j)}{\Gamma(\theta_i + \hat{\theta}_j)} s_N(\lambda) d\lambda,$$

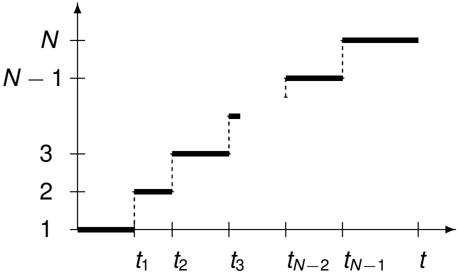
where

$$s_N(\lambda) = \frac{1}{(2\pi\iota)^N N!} \prod_{j \neq k} \Gamma(\lambda_j - \lambda_k)^{-1}$$

and the integral is along vertical lines with $\Re\lambda_i > \theta_j$ for all i, j.

Semi-discrete model

A path $\phi \equiv \{0 < t_1 < \ldots < t_{N-1} < t\}$:



$B_1, B_2, \ldots$ independent Brownian motions.

$$E(\phi) = B_1(t_1) + B_2(t_2) - B_2(t_1) + \cdots + B_N(t) - B_N(t_{N-1}).$$

$$Z_t^N(\beta) = \int e^{\beta E(\phi)} d\phi.$$

Semi-discrete model

Set $X_1^N(t) = \log Z_t^N$ and, for $k = 2, \ldots, N$,

$$X_1^N(t) + \cdots + X_k^N(t) = \log \int e^{E(\phi_1)+\cdots+E(\phi_k)} d\phi_1 \ldots d\phi_k,$$

where the integral is over non-intersecting paths $\phi_1, \ldots, \phi_k$ from $(0,1), \ldots, (0,k)$ to $(t, N-k+1), \ldots, (t, N)$.

Theorem (O'C 2009)

The process X^N is a diffusion process in $\mathbb{R}^N$ with generator

$$\frac{1}{2}\Delta + \nabla \log \psi_0 \cdot \nabla$$

where $\psi_0(x)$ is the ground state eigenfunction of the quantum Toda lattice Hamiltonian

$$H = \Delta - 2\sum_{i=1}^{N-1} e^{x_{i+1}-x_i}.$$

The function $\psi_0(x) = \Psi_0(e^x)$ is a $GL(N,\mathbb{R})$-Whittaker function. This diffusion is the analogue of Dyson's Brownian motion in this setting; the distribution of X_t^N is the analogue of the GUE.

The law of the partition function

Corollary

For $s > 0$,

$$Ee^{-sZ_t^N} = \int s^{\sum \lambda_i} \prod_i \Gamma(-\lambda_i)^N e^{\frac{1}{2}\sum_i \lambda_i^2 t} s_N(\lambda) d\lambda,$$

where

$$s_N(\lambda) = \frac{1}{(2\pi\iota)^N N!} \prod_{j \neq k} \Gamma(\lambda_j - \lambda_k)^{-1}$$

and the integral is along vertical lines with $\Re\lambda_i < 0$ *for all* i.

Connection with random matrices

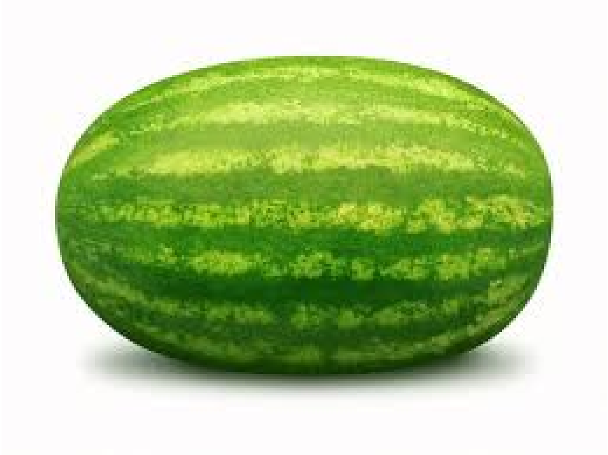
The probability measure on $\iota\mathbb{R}^N$ with density proportional to

$$e^{\sum_i \lambda_i^2 t/2} s_N(\lambda) \equiv \frac{1}{(2\pi\iota)^N N!} e^{\sum_i \lambda_i^2 t/2} \prod_{i>j} (\lambda_i - \lambda_j) \prod_{i<j} \frac{\sin \pi(\lambda_i - \lambda_j)}{\pi}$$

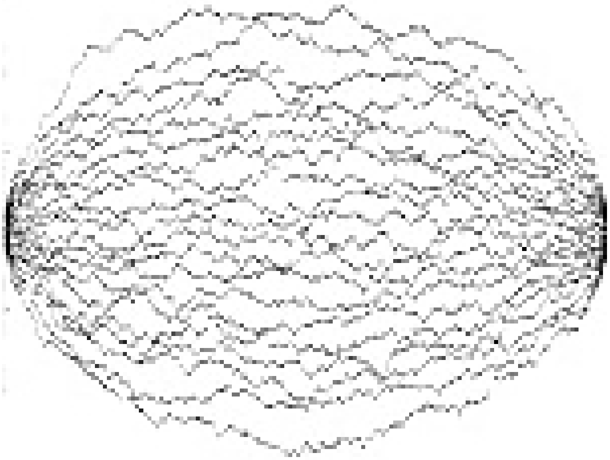
is (up to a factor of $\iota\pi$) the law, at time $1/t$, of the radial part of a Brownian motion in the symmetric space of positive definite Hermitian matrices. In particular, it is a determinantal point process, so $Ee^{-sZ_t^N}$ can be written as a Fredholm determinant.

Using a different Fredholm determinant representation, Borodin and Corwin (2011) and Borodin, Corwin and Ferrari (2012) have recently shown that $\log Z_N^N$ (rescaled) converges in distribution to the Tracy-Widom F_2 distribution.

A watermelon



A (Brownian) watermelon



Watermelons and white noise

O'C-Warren (2011): For $n = 1, 2, \ldots$, $t \geq 0$ and $x, y \in \mathbb{R}$, define

$$Z_n(t,x,y) = \\ 1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k^{(n)}((t_1,x_1),(t_2,x_2),\ldots,(t_k,x_k)) \\ \times \xi(t_1,x_1)\xi(t_2,x_2)\cdots\xi(t_k,x_k)dt_1dx_1\cdots dt_kdx_k,$$

where $\Delta_k(t) = \{0 < t_1 < \cdots < t_k < t\}$ and $R_k^{(n)}$ is the k-point correlation function of a 'watermelon': that is, a collection of n non-intersecting Brownian bridges which all start at x at time 0 and all end at y at time t.

Watermelons and white noise

Theorem (O'C-Warren 2011)

The series defining $Z_n(t, x, y)$ is convergent in $L_2(\xi)$.

This is proved by showing that $Ee^L < \infty$, where L is the total intersection local time between two 'watermelons'.

Wronskian representation

Replace ξ by smooth potential.

Theorem (O'C-Warren 2011)

$Z_n = c_{n,t} p^{-n} \tau_n$ where

$$\tau_n = \det \left[\partial_x^i \partial_y^j u(t,x,y) \right]_{i,j=0}^{n-1} \qquad c_{n,t} = t^{n(n-1)/2} \prod_{j=1}^{n-1} j!$$

The proof is via a generalisation of the Karlin-McGregor formula.
This formula also holds (in a weak sense) in white noise setting.

Coupled heat equations

Let $u_n = pZ_n/Z_{n-1}$, where $Z_0 = 1$.

Theorem (O'C-Warren 2011)

The u_n's satisfy

$$\partial_t u_n = \frac{1}{2}\partial_y^2 u_n + [\xi(t,y) + \partial_y^2 \log Z_{n-1}]u_n$$

with initial conditions $u_n(0,x,y) = \delta(x-y)$.

Coupled transport equations

Let $a_n = \tau_{n+1}\tau_{n-1}/\tau_n^2$, where $\tau_0 = 1$.

Theorem (O'C-Warren 2011)

The a_n's satisfy

$$\partial_t a_n = \frac{1}{2}\partial_y^2 a_n + \partial_y[a_n \partial_y \log u_n].$$

These equations appear to also make sense in the white noise setting (cf. Hairer 2011).

Markovian evolutions

Let $q_n = \log \tau_n - \log \tau_{n-1}$. The above evolution equations strongly suggest that, for each n, and fixed x,

$$q_1(t, x, \cdot), \ldots, q_n(t, x, \cdot)$$

is a Markov process. We prove it for $n = 2$.

Markovian evolutions

Approximation by the discrete models indicates that, for fixed t, x,

$$q_1(t,x,y), q_2(t,x,y), \ldots \qquad y \in \mathbb{R}$$

is a diffusion process in $\mathbb{R}^{\mathbb{N}}$ ($\sim$ 'semi-infinite' quantum Toda chain)

Some recent progress towards understanding this process has been made by Moreno Flores and Quastel (in prep), Corwin and Hammond (in prep), Borodin and Corwin (2011).

For large t it should rescale to the multi-layer Airy process.

2D Toda equations

Theorem (O'C-Warren 2011)

The q_n's satisfy the 2D Toda equations

$$\partial_{xy} q_n = e^{q_{n+1}-q_n} - e^{q_n-q_{n-1}}.$$

Thank you!