Vote revelation: empirical characterization of scoring rules

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Abstract

In this paper I consider choice correspondences defined on an extended domain: the decisions are assumed to be taken not by individuals, but by committees and, in addition to the budget sets, committee composition is observable and variable. For the case of varying committees choosing over a fixed set alternatives I provide a full characterization of committee choice structures that may be rationalized with sincere scoring (which has a natural interpretation if the number of alternatives is equal to 2). For the general case of variable budget sets a necessary implication of choice by sincere scoring is provided.

1 Introduction

Consider an observer trying to make sense of the goings on in a secretive committee, such as the old Soviet Politburo. Such an observer would not

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have any direct evidence about preferences of individual committee members, nor would he be likely to observe the rules the committee uses to make its decisions. Nevertheless, our Kremlinologist does have some information to work with. For one, he may have a reasonably good idea of the options the committee members are facing. He would also be able to observe the committee decision: perhaps, it would come out in the Pravda. Finally, the committee membership is public knowledge (he could determine it by observing the figures standing on the observation deck of Lenin's Mausoleum during the Revolution Day parade). What sort of deductions would it be possible to make about the unobservable preferences and preference aggregation rules within the committee from this information?

Alternatively, we may be observing the decisions of a relatively transparent group, which has formal decision rules that are explicitly set in a published law. Still, many things may be unknown about what happens inside the doors and inside the minds of the group members. Do they vote strategically or sincerely? Do they take into account preferences of and/or information possessed by their fellow committee members? If only committee decisions are made public, with votes and deliberations remaining secret, could we still test theories about the goings on inside the committee?

In fact, not much could be said from a single observation of the committee decision alone. However, as I try to establish in this paper, it turns out that, if a number of observations of decisions taken by a committee with variable membership is available, one can use the available data to test certain hypotheses about the committee functioning.

The approach I use here is, in fact, quite standard, being based on the ideas of revealed preference and rationalizability, that have long been standard foundations of economic analysis. Ever since Houthakker (1950) it has been known that a simple consistency condition on choices (the Strong Axiom of Revealed Preference, SARP) is a necessary and sufficient condition for being able to explain individual choices with rational preference maximization. Of course, this approach has long been a basis for the formal decision theory used by political scientists, as well as economists. The standard textbook treatments of the political theory, such as Austin-Smith and Banks (1999), in fact, start by formally presenting it. That observations of of group decisions themselves may be used to uncover both individual preferences and group decision rules is, however, frequently ignored.

It is not to say, that this has never been suggested. Thus, for instance, when Blair *et al.* (1976) characterized such restrictions on choice structures as would derive from maximizing preferences that are merely acyclic, rather than transitive, this could, of course, be interpreted as characterizing choices made by committees of rational members with some of those members exercising veto power. In a different context, Peters and Wakker (1991) have discussed empirical consequences of bargaining solutions, as do, for instance, Chambers and Echenique in a recent paper (2011)[1]. The characterization of the empirical consequences of such behavior for the household demands by Browning and Chiappori (1998) has been particularly influential in actual empirical work. However, though well-established, the tradition of revealed preference approach to group decisions has not been much developed recently. Thus, I am aware of no studies establishing "signatures" imposed on collective decisions by most commonly used voting rules. It is precisely this that I attempt to do in this paper.

In fact, when in recent years concepts of choice and revealed preference have received substantial renewed attention in economics, it was mostly in the context of individual decision-making. This attention has been derived from the new focus on "boundedly rational" decision-making procedures different from the usual rational preference maximization. In this context one might mention, among many others, Manzini and Mariotti (2007) work on "sequential rationalizability" or Masatlioglu and Ok (2005) study of choice with status-quo bias, both of which attempt to establish restrictions imposed on choices by distinct decision-making procedures. Other recent studies, such as Caplin and Dean (2009) and Caplin, Dean and Martin (2011) attempt to explore the restrictions that various "boundedly rational" procedures would impose on records that are somewhat more detailed than the usual choice data, though still plausibly observable: the insight that is also of key importance to the present research.

Perhaps most obvious parallel to the present study in the new behavioral literature is presented by the "multi-self" models of decision-making, such as Kalai et al. (2002), and Ambrus and Rozen (2011). In fact, Ambrus and Rozen's (2011) approach to individual behavior as resulting from aggregation of multiple utility functions (representing preferences of different "selves" in one's mind) is formally analogous to the preference aggregation in a committee (with each of the "selves" corresponding to a committee member).

[1] Notably, in that latter paper the authors demonstrate that, unless the threat point data is available and variable, the variation on the size of the surplus is not sufficient to distinguish major bargaining solutions.

As they observe, without any restrictions on the number of utility functions inside one's mind, most aggregation rules would not provide falsifiable restrictions on the resulting choices. However, if the cardinality of the set of "selves" were to be known, such restrictions would, indeed, emerge. In a way, if armed with their results the Kremlinologist of my example could have attempted to infer the lower bound on the number of Politburo members whose views had actual impact on its choices!

The key distinction between Ambrus and Rozen (2011) and the present research is, in fact, in the sort of the data that is available for testing the theory. Whereas the "selves" inside one's mind are not directly observable and may only be inferred from individual behavior, in the context of collective decisions the group membership might itself reasonably constitute part of the observed data. Not only that, but it may be natural to assume that one could observe variation not only of the set of available alternatives, as is standard in the revealed preference literature, but also of group composition itself: my Kremlinologist observes the Mausoleum's deck and could see, who is "in", and who is "out" (in another setting, for instance, we may observe decisions on a given issue of different parliamentary committees and subcommittees, the membership of which is known).

To sum up, I am going to deal with situations in which the group decision data is, in fact, richer than would be usual in revealed preference models of individual choice: in addition to the record of choices from a given set of alternatives, one has the committee membership at each decision point to consider. Hence, if one wants to test a given theory of how the committee works, one has more information to base it on. In the language of individual decision theory, group membership becomes a frame (as, for instance in Salant and Rubinstein 2008) Even if such data is incomplete (i.e., when not all possible observations might be in the data), one may hope that preference aggregation rules might have additional testable implications.

In fact, in this study I concentrate on just one particular class of theories about the internal committee workings. I will generally assume that each committee consists of rational members who make a joint decision by using some scoring rule: that is, one of the voting rules, such as the simple "first past the post" plurality or the Borda Count, in which individuals are asked to provide each alternative with a numeric score (reflecting their preferences), the individual scores are added up and the alternative with the highest aggregate score is chosen. These rules have long been characterized by social choice theorists (see Smith 1973, Young 1975 or Myerson 1995). My objective is to formulate the natural restrictions on observations implied by these rules. Even when the particular scoring rule is unknown, such restrictions turn out to be non-trivial.

The objective of this work is clearly related to the study of empirical content of sincere (vs. strategic) voting by Degan and Merlo (2009). In fact, as suggested at the start of this introduction, if the formal decision rule is known, this work may be reinterpreted precisely as the test of voter sincerity: if I know how the votes are counted, violations of the conditions established here could only be interpreted as indications that the scores do not directly reflect rational individual preference. Thus, to the extent one maintains the assumption that voters are rational, sincere voting would be falsified in this case. Likewise, this paper is related to Kalandrakis (2010) work on rationalizing individual voting decisions. This paper crucially differs from both Degan and Merlo (2009) and Kalandrakis (2010), however, in that I do not assume observability of individual votes (nor do I impose anything in addition to rationality on individual preferences).

Rather, individual votes are "revealed" here from the observations of the group choices. One hopes to characterize the conditions under which these revealed scores are consistent. For the simple case of varying committees choosing over a fixed pair of alternatives, in fact, such characterization turns out to coincide with the conditions for the existence of additive probability measures over a finite state space, representing a given binary relation "at least as likely as", established by Kraft et al. (1959). The necessary and sufficient condition for such a representation has a clear "SARP-like" acyclicity interpretation. This condition formally generalizes for any choice structure over a fixed finite set of alternatives (though the interpretation of the resultant condition may be harder). For the more general case of variable budget sets the natural necessary conditions of the "SARP"-type emerge, though the complete characterization is, so far, unknown.

The rest of this paper is organized as follows. In section two I provide the basic model set-up. In section three I consider the simple case of two alternatives and provide a characterization of the restrictions on the committee choice structures that make them consistent with choice by scoring In section four I extend the analysis to the case of three or more alternatives. Section five concludes.

2 Basic Set-up

In this section I closely follow my earlier note (Gomberg 2011). Consider a finite set $N = \{1, 2, ...n\}$ of agents and a finite set $X = \{x_1, x_2...x_m\}$ of alternatives. A set of alternatives to be considered by a *committee* $S \in 2^N \backslash \{\varnothing\}$ is $B \in 2^X \backslash \{\varnothing\}$; following the standard terminology of individual choice theory, I shall call B the *budget set*. If a committee S is offered a choice from the budget set B the committee choice is recorded as $\varnothing \neq C(B, S) \subset B$. The *committee choice structure* is defined as a pair $(\mathcal{E}, C(.,.))$ where $\mathcal{E} \subset 2^X \backslash \{\varnothing\} \times 2^N \backslash \{\varnothing\}$ is the record of which budget sets where considered by which committees and $C : \mathcal{E} \to X$, such that $C(B, S) \subset B$ is the non-empty-valued choice correspondence, recording committee choices.

In order to explain observed committee choice structures I shall, in general, assume that each agent $i \in N$ has rational (complete and transitive) preferences $\succsim_i$defined over X. The committee choice structure provides a record of observed committee choices, which may be used by an observer to deduce the preference profiles and the preference aggregation rules the committee uses. In this paper I concentrate on a particular class of such rules: the scoring rules, a class that includes such distinct procedures as the plurality vote (in which the winner is an alternative that is chosen by the largest number of voters), the Borda Count (in which alternatives get assigned the most points for being someone's top choice, a point less for being a second choice, etc., the scores get summed up over all the voters and the alternative with the largest score wins), or the Approval Voting (in which an individual is allowed to mark alternatives as acceptable or unacceptable, and the alternative which has been marked as acceptable by the largest number of voters gets chosen).

In general, I shall assume that agents are non-strategic, in that they ignore who else is in the committee (as noted above, the conditions I am deriving here might, if the formal rule is observable, be viewed as empirical implications of sincere voting itself). However, I shall allow the votes to depend on the budget sets under consideration (as would be the case in a sincere Borda Count). Thus, if the set of alternatives B, a vote of agent $i \in S$ is a function $v_i^B : B \to \mathbb{R}$. Vote independence from committee membership is, in fact, an extremely strong condition that not only would be inconsistent with strategic voting, but would also eliminate possible vote variations due to interdependent preferences or differential information.

Given a vote from each of its members a committee S chooses an alternative that gets the highest score

$$C^{scoring}(B,S) = \arg\max_{x\in B} \sum_{i\in S} v_i^B(x)$$

where $\sum_{i\in S} v_i^B(x)$ is called the *score* received by an alternative $x \in B$ in voting by committee S. Such a choice structure is said to be generated by the scoring rule.

Following Myerson (1995) I shall allow agents to submit votes that are distinct from reporting their preference orderings. In fact, for the purposes of defining a scoring rule one does not need to assume that the votes themselves derive from rational preferences. All the scoring rules require agents is to report a ranking of alternatives in B by means of their votes $v_i \in \mathbb{R}^k$. In general such a ranking may not necessarily represent a rational preference (and thus, for instance, could be inconsistent over the different budget sets B). Nevertheless I shall concentrate on voting that, indeed, can be viewed as a sincere representation of individual preferences. Formally, given a rational preference profile $\succsim = (\succsim_1, \succsim_2, ..., \succsim_n)$ I shall say that a committee vote v_i^B is *strictly consistent with preferences* if $x \succsim_i y$ if and only if $v_i^B(x) \geq v_i^B(y)$. I shall say that a committee vote v_i^B is *weakly consistent with preferences* if $x \succsim_i y$ implies $v_i^B(x) \geq v_i^B(y)$. The obvious reason why both consistency notions are of interest here is that though in some scoring rules agents could, in fact allow agents to submit what amounts to utility functions (i.e., actual representations of their preferences), other rules do not. Thus, a "sincere Borda Count" would be strictly consistent with preferences, while "sincere plurality" would only be weakly consistent as long as there are at least 3 alternatives.

If a committee choice structure is such that for any $(B,S) \in \mathcal{E}$

$$C(B,S) = C^{scoring}(B,S)$$

where the votes are consistent with preferences for some rational preference profile $\succsim$. I shall say that $\succsim$ rationalizes $(\mathcal{E}, C(.,.))$ via a scoring rule.

It should be noted, that unless the choice structure is extended by allowing observing variations in committee membership, scoring rules would, at first glance, appear particularly unpromising from the standpoint of this research: it would seem that nearly every possible committee decision could be explained by some sort of scoring applied to an unobserved preference profile of a fixed committee. Thus, if one defines, in the spirit of Salant and Rubinstein (2008) work on the choice with frames, the choice correspondence as

$$C(B) = \{x : x \in C(B, S) \text{ for some committee } S\}$$

not much structure appears be imposed on $C_c(.)$, though it would follow from Ambrus and Rozen (2011) that some restrictions may be derived from the cardinality of N, if that is observed. However, it turns out that more can be said if committee membership and its variations are observed.

3 Revealed Scoring: The case of two alternatives

I shall first consider the simple case, in which the number of alternatives is equal to 2. In this case, scoring may be interpreted as weighted majority vote in which, for the purposes of this model, both the individual preferences and weights are unobservable. In this case the only interesting budget set is $B = X = \{x_1, x_2\}$, so that the entire observable variation comes from the committee membership. The choice C here is a mapping from a subset of the set of observed committees $\mathcal{E} \subset 2^N \setminus \{\varnothing\}$, that may take one of only three values: $\{x_1\}$, $\{x_2\}$ or $\{x_1, x_2\}$.

It is clear that not every such committee choice structure would be rationalizable with sincere scoring. Crucially, the notion of sincere scoring studied here implies that each individual's votes are independent of the committee composition. Hence, if we ever observe that for two disjoint committees $S \cap T = \varnothing$ we have $C(S) = C(T) = x_i$ it must, indeed, follow that $C(S \cup T) = x_i$. This, property, introduced, for instance, in characterizations of scoring rules by Smith (1973) and Young (1975) is usually known as the *reinforcement* axiom. Clearly, reinforcement must be a necessary condition for the rationalizability here desired. But the scoring has an even stronger implication for the actual scores that committees assign to alternatives: the score difference between the alternatives must be added up if two disjoint committees are joined.

In fact, if sincere scoring is the rule used, the difference w between the scores assigned to x_1 and to x_2 by the committee S

$$w(S) = \sum_{i \in S} v_i^B(x_1) - \sum_{i \in S} v_i^B(x_2)$$

will define a (signed) measure on the finite measurable space $\left(N, 2^N\right)$, as long as one naturally sets $w(\varnothing) = 0$, since $w(S \cup T) = w(S) + w(T) - w(S \cup T)$ for any two committees $S, T \in 2^N$.

Unfortunately, we do not observe the actual scores or their differences, but only choices, which correspond to the sign of w. Defining $\mathcal{E}^* = \mathcal{E} \cup \varnothing$ it may be convenient to summarize our observations with a function $f : \mathcal{E}^* \to \{-1, 0, 1\}$ defined by the

$$f(S) = sign\,(w(S)) = \begin{cases} -1, \text{ if } C(S) = \{x_2\} \\ 0, \text{if } C(S) = \{x_1, x_2\} \text{ or } S = \varnothing \\ 1, \text{if } C(S) = \{x_1\} \end{cases}$$

This function f is, of course, non-additive. If, however, we can consistently with it assign individual vote differences w_j to each individual so that the committee differences defined as

$$sign\,(w(S)) = sign\left(\sum_{j \in S} w_j\right) = f(S)$$

, we shall obtain a scoring-based theory that would explain how the observed choice structure arose!

Fortunately, it turns out that this problem is closely related to well-established problems in utility theory. In fact, a very similar mathematical problem emerges if one considers the question of when could a binary relation "at least as likely as" over a finite states space be represented by a probability measure, which has been posed and solved by Kraft et al. (1959). Indeed, the following example they construct implies that the reinforcement alone, though necessary, is not sufficient for such a theory to be possible.

Example 1 *Suppose* $N = \{1, 2, 3, 4, 5\}$ *and* $f(\{4\}) = f(\{2, 3\}) = f(\{1, 5\}) = f(\{1, 3, 4\}) = 1$ *whereas* $f(\{1, 3\}) = f(\{1, 4\}) = f(\{3, 4\}) = f(\{2, 5\}) = -1$. *It can be checked that it is possible to complete this set of observed choices in a way that would not violate reinforcement. However, it is not hard to see that this set of choices is not consistent with sincere scoring, as it would imply that* $2w_1 + w_2 + 2w_3 + 2w_4 + w_5$ *is simultaneously positive and negative!*

Consequently, a stronger condition, which I shall call *strong reinforcement,* is required, which is analogous to strong additivity of Kraft et al.

(1959). Following Fishburn (1986) it can be presented as follows. Consider two collections (of equal cardinality) of committees $\mathbf{S} = (S_1, S_2, ..., S_m)$ and $\mathbf{T} = (T_1, T_2, ..., T_m)$. Note, that an empty set is taken here as a possible committee and that a committee might be repeated several times within a collection. Denote as $n_j(\mathbf{S})$ the number of committees in the collection $\mathbf{S}$ that individual j is included in. We say that $\mathbf{S} \approxeq \mathbf{T}$ if for each individual $j \in N$ $n_j(\mathbf{S}) = n_j(\mathbf{T})$.

- The choice correspondence C satisfies *strong reinforcement* if for each pair of committee collections $\mathbf{S}, \mathbf{T}$ such that $\mathbf{S} \approxeq \mathbf{T}$ if $f(S_i) > f(T_i)$ or $f(S_i) = f(T_i) = 0$ for $i = 1, 2, ..., m-1$ then not $f(S_m) > f(T_m)$.

It should be noted that strong reinforcement is indeed a strong property, which implies a number of desirable conditions of the choice structures. Thus, it can be easily seen to imply the reinforcement property itself. It turns out that, in fact, this condition characterizes choice structures that can be explained with sincere scoring.

Theorem 1 *A committee choice structure $(\mathcal{E}, C(.,.))$ may be generated by a scoring rule strictly consistent with rational preferences if and only if the choice structure satisfies strong reinforcement.*

Proof. The necessity part is straightforward, since if it were not the case, there would exist a pair of committee collections $\mathbf{S} \approxeq \mathbf{T}$ such that $f(S_i) > f(T_i)$.or $f(S_i) = f(T_i) = 0$ for all $i = 1, 2, ..., m-1$ and $f(S_m) > f(T_m)$ However, as $f(S_i) = sign(w(S_i)) = sign\left(\sum_{j \in S_i} w(\{j\})\right)$ it follows that $\sum_{j \in S_i} w_j > \sum_{j \in T_i} w_j$ or $\sum_{j \in S_i} w_j = \sum_{j \in T_i} w_j = 0$ for $i = 1, 2, ...m-1$ and $\sum_{j \in S_m} w_j > \sum_{j \in T_m} w_j$, which, if we some across the committees in each collection, in turn would imply that $\sum_{j \in N} n_j(\mathbf{S}) w_j > \sum_{j \in N} n_j(\mathbf{T}) w_j$.

The proof of sufficiency closely follows that of Theorem 4.1 in Fishburn (1970). If all committees make the same choice, the theorem is trivially true, therefore, I shall henceforth assume that there exists at least one pair of committees $(S, T) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $f(S) > f(T)$ Let $K \in \mathbb{N}$ be equal to the number of distinct committee pairs $S, T \in \mathcal{E}$ $(S \neq T)$ such that $f(S) > f(T)$ and $M \in \mathbb{Z}_+$ be equal to one half of the number of committee pairs $(S,T) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $f(S) = f(T) = 0$. Note that the latter includes committee pairs of the form $(S,\varnothing)$ and $(\varnothing,T)$. Clearly, $K + M \leq 2^n < \infty$.

For each committee S let the indicator function

$$1_S(j) = \begin{cases} 1 \text{ if } j \in S \\ 0 \text{ if } j \notin S \end{cases}$$

Clearly, if for each of the first $k = 1, 2..., K$ committee pairs S^k, T^k defined above we may write

$$\sum_{j=1}^{n} w_j a_j^k > 0$$

and for each of the following $k = K+1, K+2, ...K+M$ committee pairs S^k, T^k we may write

$$\sum_{j=1}^{n} w_j a_j^k = 0$$

where $a_j^k = (1_{S^k}(j) - 1_{T^k}(j)) \in \{-1, 0, 1\}$, the weights $\sum_{j=1}^{n} w_j$ may be interpreted as a "reconstruction" of the individual vote difference consistent with the observed choice structure (note, in particular, that this would imply that $\sum_{j=1}^{n} w_j 1_S(j) = 0$ for every S such that $f(S) = 0$)

Suppose this is impossible. Then by Theorem 4.2 in Fishburn (1970), know as the Theorem of the Alternative, there must exist a collection of numbers $r_{k,}$ $k = 1, 2, ..., M+K$, such that the first K of these are non-negative and not all zero so that for every $j = 1, 2, ..., n$

$$\sum_{k=1}^{K+M} r_k a_j^k = 0$$

In fact, since all a_j^k are rational by construction, all r_k may be chosen to be integers. If for some $k > K$ there is an $r_k < 0$ one may replace a_j^k with $-a_j^k$ to make it positive (this is possible since if $f\left(S^k\right) = f\left(T^k\right)$ one may interchange S^k and T^k). Consider now two committee collections $\mathbf{S}$ and $\mathbf{T}$ such that each committee S^k is repeated r^k times in $\mathbf{S}$ and each committee T^k is repeated r^k times in $\mathbf{T}$. By construction the cardinality of each committee collection is equal to $\sum_{k=1}^{K+M} r_k$ and from the preceding equation it follows that the number of times each individual is included in committees in each collection is

$$n_j(\mathbf{S}) = \sum_{k=1}^{K+M} r_k 1_{S^k}(j) = \sum_{k=1}^{K+M} r_k 1_{T^k}(j) = n_j(\mathbf{T})$$

and, hence $\mathbf{S} \approxeq \mathbf{T}$. But by construction we have $f\left(S^k\right) \geq f\left(T^k\right)$ for all $k = 1, 2, ...K + M$, with the first K inequalities strict. Hence, the strong reinforcement of the committee choice structure is violated. QED

4 Three or more alternatives

4.1 Constant budget set

If there are three or more alternatives the problem cannot be reduced to that of an existence of a single measure on the committee space. Nevertheless, as long as all the committees are facing the same choice problem (i.e., the budget set B is not varied), the linear structure of the scoring rules utilized in the previous section allows for a very similar formulation.

Our basic objective remains the same: to find vote scores for each individual that would explain the observed committee choices. Notably, once there are at least three alternatives, we now will have to avoid "scoring cycles", as the following example shows.

Example 2 *Consider the budget set $B = \{a, b, c\}$ and the four disjoint committees S_1, S_2, S_3 and T. Let $C(B, S_1) = a$, $C(B, S_2) = b, C(B, S_3) = c$, $C(B, S_1 \cup T) = b, C(B, S_2 \cup T) = c, C(B, S_3 \cup T) = a$. It is not hard to see that this implies that $bP_{B,T}cP_{B,T}aP_{B,T}b$ which, of course, implies and impossible cycle: committee T should be giving alternative b a higher score than alternative c, alternative c a higher score than alternative a, and alternative a the higher score than alternative b, which is impossible.*

As the example above suggests, the scores may be "revealed" through observed committee choices (see Gomberg 2011). As, for the rest of this section, the budget set is fixed, I shall only consider variations in the committee membership $S \subset N$.

- *Direct revelation.* For each $S \in \mathcal{E}$ a pair of nested binary relations $P_S^* \subset R_S^*$ on B is defined by

(i) let $x \in C(B, S)$ then $xR_S^* y$ for any $y \in B$

(ii) let $x \in C(B, S)$ and $y \notin C(B, S)$ for some $y \in B$ then $xP_S^* y$

This constitutes a record of direct preference revelation: if an alternative is chosen, it implies it received at least as high a score as any other feasible alternative and a strictly higher score than any feasible alternative not chosen.

Consider the total set of observations we have. If our theory is correct and this choice is rationalized with scoring, in the actual vote count each observation of $xP_S^* y$ it must have been obtained from $\sum_{i \in S} v_i^B(x) > \sum_{i \in S} v_i^B(y)$ and each $xR_S^* y$ from $\sum_{i \in S} v_i^B(x) \geq \sum_{i \in S} v_i^B(y)$. These are, of course, linear inequalities. In fact, the set of all "revealed scoring" statements must have been generated by a system of linear inequalities, which would have to hold simultaneously for the rationalization to be possible.

Let the cardinality $\#B = m$. For simplicity, for the moment I shall restrict myself to the single-valued choice correspondences (the extension to the mutli-valued choice correspondences is straightforward). Consider a vector $w = (w_1, w_2, ..w_n, w_{n+1}, ...w_{2n}, ...w_{nm}) \in R_+^{nm}$ where w_{kn+j} corresponds to the reconstructed vote that agent k emits for alternative m. As in the previous subsection, I shall consider each revealed scoring statement (taking care to track the committee by which it has been generated). As the total number of such statements is finite, let K be the number of strict statements $xP_S^* y$ and M be one half of the rest.

Consider a list of all such revealed scoring pairs If the kth pair is $x_k P_S^* x_l$ (for the first K elements of the list) or $x_p R_S^* x_r$ (for the rest) then one can define $a_j^k = 1$ for all $j = p + ms$, where $s \in S$ $a_j^k = -1$ for all $j = r + ms$, where $s \in S$, and $a_{ij} = 0$ otherwise. As in the case of two alternatives, if for each of the first $k = 1, 2..., K$ revealed preference scoring relations defined above we may write

$$\sum_{j=1}^{n} w_j a_j^k > 0$$

and for each of the following $k = K + 1, K + 2, ...K + M$ revealed scoring relations we may write

$$\sum_{j=1}^{n} w_j a_j^k = 0$$

we would rationalize the observed choice structure.

As in the previous section, the Theorem of the Alternative allows one to restate the problem of existence of a solution to this system of inequalities as a problem of existence of a solution to the equation

$$\sum_{k=1}^{K+M} r_k a_j^k = 0 \qquad (*)$$

where $(r_1, r_2, ...r_{K+M}) \in \mathbb{Z}^{K+M}$ with the first K terms non-negative and not all equal to zero.

As in the case of two alternatives, this condition is, in fact, necessary and sufficient for the existence of rationalization by scoring, though it is harder to get its intuitive interpretation. A greater feeling for its implication may be obtained if we reformulate a necessary implication of it in a more familiar "revealed preference" form.

Consider, for instance, the "indirect revealed scoring" implied by the reinforcement property of the scoring rules (which, as noted above states that if two disjoint committees make the same choice from a given budget set, so should their union). We can then define the following .

- *Reinforcement*[2]

  The binary relations $P_S \subset R_S$ on B are defined by

  (i) xP^*y implies xPy, xR^*y implies xRy,

  (ii) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \cap T = \varnothing$, xR_Sy and xR_Ty imply that $xR_{S\cup T}y$

  (iii) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \cap T = \varnothing$, $xP_{B,S}y$ and $xR_{B,T}y$ imply that $xP_{S\cup T}y$

  (iv) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \subset T(T\backslash S \neq \varnothing)$, xP_Sy and yR_Tx imply that $yP_{T\backslash S}x$

  (v) For any $S, T \in 2^N \backslash \{\varnothing\}$ such that $S \subset T(T\backslash S \neq \varnothing)$, xR_Sy and yP_Tx imply that $yP_{T\backslash S}x$

With this in mind we may now define a simple acyclicity condition, motivated by the example above:

[2]Note that example 1 above shows that a stronger indirect extension could be imposed here. However, reinforcement is more intuitive, so I stick to it as a necessary implication of rationalizability.

Axiom 1 (Committee Axiom of Revealed Preference (CARP)) [3]

For any $S \in 2^N \setminus \{\varnothing\}$ *and any* $x_1, x_2, ...x_n \in B$,
$x_1 R_S x_2, x_2 R_S x_3 ... x_{n-1} R_S x_n$ *implies* $\neg (x_n P_S x_1)$

It is straightforward to see that CARP is, in fact, implied by scoring

Proposition 1 *A committee choice structure* $(\mathcal{E}, C(.,.))$ *may be generated by a scoring rule strictly consistent with rational preferences only if the implied* R_S *and* P_S *satisfy CARP for each* $S \in 2^N \setminus \{\varnothing\}$.

4.2 Variable budget sets: Necessary conditions

If the budget sets are variable, however, there is no obvious way of restating this as a solution to a linear system. The problem is that reconstructed scores may vary depending on the budget set in question (think of the number of points an alternative gets from a given individual under Borda Count in different budget sets), but such scores should still be consistent with an underlying rational preference. Maintaining the requirement of strict consistency of votes with an underlying rational preference relation, I shall try to "reveal" as much as possible about such individual votes.

Naturally, the condition (*) would still have to hold for each budget set, but it is no longer sufficient for rationalizabitliy with scoring. A further necessary implication may be obtained by using the direct score revelation and its extension by reinforcement defined above.

However, the budget set variation provides us with further information that allows one to make inferences about individual preferences from either direct or indirect observations of singleton coalitions. If the agents votes are strictly consistent with underlying preferences, then if an individual is every revealed to give a greater score to one alternative than to another, this should be maintained in all budget sets. I will therefore extend the revealed score relations and define the individual revealed preference relation P_i as follows:

- *Individual preference revelation for strictly consistent scoring*

  (i) If $xR_{B,\{i\}}y$ for some $B \in 2^X \setminus \{\varnothing\}$ then $xR_{D,\{i\}}y$ for any D s.t. $x, y \in D$ and xR_iy

  (ii) if $xP_{B,\{i\}}y$ for some $B \in 2^X \setminus \{\varnothing\}$ then $xP_{D,\{i\}}y$ for any D s.t. $x, y \in D$ and xP_iy

[3] The naming suggestion for this axiom belongs to Norman Schofield

Once the binary relations $R_{B,S}$ and $P_{B,S}$ are thus extended (including further extensions by reinforcement, if possible), we may formulate a stronger version of CARP:

Axiom 2 (CARP*)

For any $B \in 2^X \backslash \{\varnothing\}$, any $S \in 2^N \backslash \{\varnothing\}$ and any $x_1, x_2, ...x_n \in B$, $x_1 R_{B,S} x_2, x_2 R_{B,S} x_3 ... x_{n-1} R_{B,S} x_n$ implies $\neg (x_n P_{B,S} x_1)$

It should be noted that, taking $B = X$, as long as individual preference revelation is taken into account, CARP implies the usual Strong Axiom of Revealed Preference (SARP) for the individual preference revelation.

It is clear that CARP would have to hold if a committee of rational individuals is deciding by sincere votes using a scoring rule, since otherwise we'd have to accept either cycles in individual preferences or in group scores (as in the example above). Hence, the next result follows immediately from the construction.

Proposition 2 *A committee choice structure $(\mathcal{E}, C(.,.))$ may be generated by a scoring rule strictly consistent with rational preferences only if the implied $R_{B,S}$ and $P_{B,S}$ satisfy CARP* for each $B \in 2^X \backslash \{\varnothing\}$ and each $S \in 2^N \backslash \{\varnothing\}$.*

The content of individual preference revelation, however, would be different if the scoring is only weakly consistent with preferences. In this case the individual preference revelation is more limited:

- *Individual preference revelation for weakly consistent scoring*

  if $x P_{B,\{i\}} y$ for some $B \in 2^X \backslash \{\varnothing\}$ then $x R_{D,\{i\}} y$ for any D s.t. $x, y \in D$ and $x P_i y$

Though this more limited extension of the revealed scoring relation may be used to define a CARP as in the case of strictly consistent scoring (I shall call it CARP**), the latter would no longer imply the SARP for individual revealed preference, which would have to be assumed directly

Axiom 3 (Strong Axiom of Revealed Preference (SARP)) *For any $i \in N$ $x_1 P_i x_2 .... P_i x_n$ implies $\neg x_n P_i x_1$*

Proposition 3 *A committee choice structure $(\mathcal{E}, C(.,.))$ may be generated by a scoring rule weakly consistent with rational preferences only if the implied $R_{B,S}$ and $P_{B,S}$ satisfy CARP** for each $B \in 2^X \backslash \{\varnothing\}$ and each $S \in 2^N \backslash \{\varnothing\}$ and P_i satisfies SARP for each $i \in N$.*

5 Conclusions and further research

This paper introduces the notion of a committee choice structure and establishes a necessary and sufficient condition for such a choice structure to be rationalizable via scoring rules when the committees decide over the fixed set budget set, with a natural interpretation for the case of two alternatives. For the case when budget sets vary, so. far it has been possible to establish a set of properties of committee choice structures that are necessary consequences of sincere scoring-based committee decisions It remains to see if this could be strengthened to a concise sufficient condition for rationalizability with scoring. An interesting further extension of the model would be to consider the consequences of particular scoring rules, such as plurality, approval or the Borda Count.

In terms of practical application, this paper provides conditions on the choice structures that would have to be violated for models more complicated than "sincere scoring" being possible to test. It should be noted that "sincerity", as defined here, is simply a statement that voters always maintain the same ranking of alternatives and do not change their scores based on the identity of other people in the committee they are a part of. Of course, if voters behave in this way, it is impossible to distinguish a possible case of "strategic" voting from simply following a fixed preference relation. However, in many environments following one's preferences would imply a violation of sincerity as here defined. Thus, for instance, if voters have interdependent preferences with other committee members, or noisy signals about a common value of different alternatives, then they may change their behavior based on the identities of other committee members. Hence, choices of committees, consisting of such voters, even if they use a scoring rule, would likely violate conditions derived here. This suggests, that the approach in this paper may be used to develop tests for presence of preference interdependence or common value in voting settings when only committee decisions and memberships are observed.

References

[1] Ambrus, A and Rozen K (2011), Rationalizing Choice with Multi-Self Models, Cowles Foundation Discussion Paper No. 1670

[2] Austen-Smith, David and Jeffrey S. Banks (1999),. *Positive Political Theory I: Collective Preference,* University of Michigan Press:1999

[3] Blair D., Bordes G., Kelly J. and Suzumura K. (1976) Impossibility Theorems without Collective Rationality, *Journal of Economic Theory* **13**: 361-379

[4] Browning, M. and Chiappori PA (1998), Efficient Intrahousehold Allocations: A General Characterization and Empirical Tests, *Econometrica* **66**: 1241-1278

[5] Caplin A. and Dean M. (2011), Search, Choice and Revealed Preference, *Theoretical Economics* **6**: 19 - 48

[6] Caplin, A., M. Dean M. and D. Martin (2011), Search and Satisficing, *forthcoming* in *American Economic Review*

[7] Chambers C., and F. Echenique (2011), Testable implications of Bargaining Theories, *manuscript*, Caltech.

[8] Ching, S (1996), A Simple Characterization of the Plurality Rule", *Journal of Economic Theory* **71:** 298-302

[9] Degan A. and Merlo A. (2009) Do Voters Vote Ideologically, *Journal of Economic Theory* **144:** 1868-1894

[10] Fishburn, PC (1970), *Utility Theory for Decision-Making*, John Wiley and Sons : 1970

[11] Fisburn PC (1986), Axioms of Subjective Probability, *Statistical Science* **1**: 335-345

[12] Gomberg A (2011), Vote Revelation: Empirical Content of Scoring Rules, in Schofield N., Caballero G., *Political Economy of Democracy and Voting*, Springer Verlag: 2011: 411-417.

[13] Houthakker HS (1950) Revealed Preference and the Utility Function, *Economica* **17**:159-174

[14] Kalai G, Rubinstein, A and Spiegler, R. (2002), Rationalizing Choice Functions by Multiple Rationales, *Econometrica* **70**: 2481-2488

[15] Kalandrakis T. (2010) Rationalizable Voting, *Theoretical Economics* **5**: 93-125

[16] Kraft CH, Pratt JW and Seidenberg A (1959) Intuitive Probability on Finite Sets, *Annals of Statistics* **30**: 408-419

[17] Manzini P. and Mariotti M. (2007) Sequentially Rationalizable Choice, *American Economic Review* **97**: 1824-1839

[18] Masatlioglu Y. and Ok E. (2005) Rational Choice with Status Quo Bias, Journ Econ Theory 121: 1-29

[19] Peters, H. and P. Wakker (1991), Independence of Irrelevant Alternatives and Revealed Group Preference, *Econometrica* **59**: 1787 - 1801

[20] Myerson R.(1995) Axiomatic Derivation of the Scoring Rules without the Ordering Assumption, *Social Choice and Welfare* **12**: 59-74

[21] Salant Y. and Rubinstein A. (2008) (A,f): Choice with Frames, *Review of Economic Studies* **75**: 1287 - 1296

[22] Smith JH (1973) Aggregation of Preferences with Variable Electorate, *Econometrica* **41**: 1027-1041

[23] Young P (1975) Social Choice Scoring Functions", *SIAM Journal on Applied Mathematics* **28**: 824 - 838