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Gapped Ground State Phases of Quantum Lattice Systems[1]

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based on joint work with

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Outline

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- Automorphic equivalence
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- Symmetry protected phases
- Spin chains with Matrix Product Ground States (MPS)
- The excess spin operators and a new invariant
- Concluding remarks

What is a quantum ground state phase?

By ground state phase we mean a set of models with qualitatively similar behavior in the ground state(s).

Concretely, this is taken to mean that a g.s. ψ_0 of one model could evolve in finite time to a g.s. ψ_1 of another model in the same phase by some physically acceptable dynamics (one generated by a short-range time-dependent Hamiltonian).

Such dynamics cannot induce or destroy long range order in finite time, and the large-scale entanglement structure remains unchanged.

In the physics literature the standard definition is that there is a curve of Hamiltonians with finite-range interactions, $H(\lambda), \lambda \in [0,1]$, such that one (or set of) ground state(s) belongs to $H(0)$ and the other to $H(1)$, and such that there is a uniform positive lower bound for the spectral gap above the g.s. for all $\lambda \in [0,1]$ (absence of a quantum phase transition).

(Quasi-local) Automorphic Equivalence

For systems in a finite volume Λ, a physically acceptable dynamics is described by a quasi-local unitary V_Λ, solution of the Schrödinger equation:

$$\frac{d}{ds} V_\Lambda(s) = iD_\Lambda(s) V_\Lambda(s), \quad s \in [0,1], \quad V_\Lambda(0) = \mathbb{1},$$

where $D_\Lambda(s)$ is a "Hamiltonian" with short-range interactions:

$$D_\Lambda(s) = \sum_{X \subset \Lambda} \Omega_s(X).$$

When we take the thermodynamic limit

$$\lim_{\Lambda \uparrow \Gamma} V_\Lambda(s)^* A V_\Lambda(s) = \alpha_s(A), \quad A \in \mathcal{A}_{\Lambda_0},$$

this dynamics converges to quasi-local automorphisms of the algebra of observables.

Ground states of quantum spin models

By quantum spin system we mean quantum systems of the following type:

- (finite) collection of quantum systems labeled by $x \in \Lambda$, each with a finite-dimensional Hilbert space of states $\mathcal{H}_x$. E.g., a spin of magnitude $S = 1/2, 1, 3/2, \ldots$ would have $\mathcal{H}_x = \mathbb{C}^2, \mathbb{C}^3, \mathbb{C}^4, \ldots$.
- The Hilbert space describing the total system is the tensor product

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x.$$

with a tensor product basis $|\{\alpha_x\}\rangle = \bigotimes_{x\in\Lambda} |\alpha_x\rangle$

- The algebra of observables of the composite system is

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x) = \mathcal{B}(\mathcal{H}_\Lambda).$$

If $X \subset \Lambda$, we have $\mathcal{A}_X \subset \mathcal{A}_\Lambda$, by identifying $A \in \mathcal{A}_X$ with $A \otimes \mathbb{1}_{\Lambda\setminus X} \in \mathcal{A}_\Lambda$. Then

$$\mathcal{A} = \overline{\bigcup_\Lambda \mathcal{A}_\Lambda}^{\|\cdot\|}$$

A common choice for Λ's are finite subsets of a graph Γ (often called the 'lattice'). E.g., if $\Gamma = \mathbb{Z}^\nu$, we may consider Λ of the form $[1, L]^\nu$ or $[-N, N]^\nu$.

Interactions, Dynamics, Ground States

The Hamiltonian $H_\Lambda = H_\Lambda^* \in \mathcal{A}_\Lambda$ is defined in terms of an interaction Φ: for any finite set X, $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$, and

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$$

For finite-range interactions, $\Phi(X) = 0$ if $\operatorname{diam} X \geq R$.
Heisenberg Dynamics: $A(t) = \tau_t^\Lambda(A)$ is defined by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$$

For finite systems, ground states are simply eigenvectors of H_Λ belonging to its smallest eigenvalue (sometimes several 'small eigenvalues').

Suppose Φ_0 and Φ_1 are two interactions for two models on lattices Γ.

Each has its set $\mathcal{S}_i$, $i = 0, 1$, of ground states in the thermodynamic limit. I.e., for $\omega \in \mathcal{S}_i$, there exists

$$\psi_{\Lambda_n} \text{ g.s. of } H_{\Lambda_n} = \sum_{X \subset \Lambda_n} \Phi_i(X),$$

for a sequence of $\Lambda_n \in \Gamma$ such that

$$\omega(A) = \lim_{n \to \infty} \langle \psi_{\Lambda_n}, A\psi_{\Lambda_n} \rangle.$$

If the two models are in the same phase, we have a suitably local automorphism α such that

$$\mathcal{S}_1 = \mathcal{S}_0 \circ \alpha$$

This means that for any state $\omega_1 \in \mathcal{S}_1$, there exists a state $\omega_0 \in \mathcal{S}_0$, such that the expectation value of any observable A in ω_1 can be obtained by computing the expectation of $\alpha(A)$ in ω_0:

$$\omega_1(A) = \omega_0(\alpha(A)).$$

The quasi-local character of α guarantees that the support of $\alpha(A)$ need not be much larger than the support of A in order to have this identity with small error.

Where do such quasi-local automorphisms α come form?

Fix some lattice of interest, Γ and a sequence $\Lambda_n \uparrow \Gamma$. Let $\Phi_s, 0, \leq s \leq 1$, be a differentiable family of short-range interactions for a quantum spin system on Γ.
Assume that for some $a, M > 0$, the interactions Φ_s satisfy

$$\sup_{x,y\in\Gamma} e^{ad(x,y)} \sum_{\substack{X\subset\Gamma \\ x,y\in X}} \|\Phi_s(X)\| + |X|\|\partial_s\Phi_s(X)\| \leq M.$$

E.g,

$$\Phi_s = \Phi_0 + s\Psi$$

with both Φ_0 and Ψ finite-range and uniformly bounded.
Let $\Lambda_n \subset \Gamma$, $\Lambda_n \to \Gamma$, be a sequence of finite volumes, satisfying suitable regularity conditions and suppose that the spectral gap above the ground state (or a low-energy interval) of

$$H_{\Lambda_n}(s) = \sum_{X\subset\Lambda_n} \Phi_s(X)$$

is uniformly bounded below by $\gamma > 0$.

Theorem (Bachmann, Michalakis, N, Sims (2012))

Under the assumptions of above, there exist a co-cycle of automorphisms $\alpha_{s,t}$ of the algebra of observables such that $\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_{s,0}$, for $s \in [0,1]$.
The automorphisms $\alpha_{s,t}$ can be constructed as the thermodynamic limit of the s-dependent "time" evolution for an interaction $\Omega(X,s)$, which decays almost exponentially.

Concretely, the action of the quasi-local transformations $\alpha_s = \alpha_{s,0}$ on observables is given by

$$\alpha_s(A) = \lim_{n\to\infty} V_n^*(s) A V_n(s)$$

where $V_n(s)$ solves a Schrödinger equation:

$$\frac{d}{ds} V_n(s) = i D_n(s) V_n(s), \quad V_n(0) = \mathbb{1},$$

with $D_n(s) = \sum_{X\subset\Lambda_n} \Omega(X,s)$.

The $\alpha_{t,s}$ satisfy a Lieb-Robinson bound of the form

$$\|[\alpha_{t,s}(A), B]\| \leq \|A\|\|B\| \min(|X|, |Y|) e^{C|t-s|} F(d(X, Y)),$$

where $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$, $d(X, Y)$ is the distance between X and Y. $F(d)$ can be chosen of the form

$$F(d) = Ce^{-b\frac{d}{(\log d)^2}}.$$

with $b \sim \gamma/v$, where γ and v are bounds for the gap and the Lieb-Robinson velocity of the interactions Φ_s, i.e., $b \sim a\gamma M^{-1}$.

Product Vacua with Boundary States (PVBS)

(with Sven Bachmann, PRB 2012, CMP to appear)

We consider a quantum spin chain with $n+1$ states at each site that we interpret as n distinguishable particles labeled $i = 1, \ldots, n$, and an empty state denoted by 0.

The Hamiltonian for a chain of L spins is given by

$$H_{[1,L]} = \sum_{x=1}^{L-1} h_{x,x+1}, \tag{1}$$

where each $h_{x,x+1}$ is a sum of 'hopping' terms (each normalized to be an orthogonal projection) and projections that penalize particles of the same type to be nearest neighbors.

$$h = \sum_{i=1}^{n} |\hat{\phi}_i\rangle\langle\hat{\phi}_i| + \sum_{1 \le i \le j \le n}^{n} |\hat{\phi}_{ij}\rangle\langle\hat{\phi}_{ij}|,$$

The $\phi_{ij} \in \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$ are given by

$$\phi_i = |i,0\rangle - e^{-i\theta_{i0}}\lambda_i^{-1}|0,i\rangle\,, \phi_{ij} = |i,j\rangle - e^{-i\theta_{ij}}\lambda_i^{-1}\lambda_j|j,i\rangle\,, \phi_{ii} = |i,i\rangle$$

for $i = 1, \ldots, n$ and $i \neq j = 1, \ldots, n$.
The parameters satisfy: $\theta_{ij} \in \mathbb{R}$, $\theta_{ij} = -\theta_{ji}$, and $\lambda_i > 0$, for $0 \le i,j \le n$, and $\lambda_0 = 1$.

There exist $n+1$ $2^n \times 2^n$ matrices $v_0, v_1, \ldots, v_n$, satisfying the following commutation relations:

$$v_i v_j = e^{-i\theta_{ij}} \lambda_i \lambda_j^{-1} v_j v_i, \quad i \neq j \tag{2}$$

$$v_i^2 = 0, \quad i \neq 0 \tag{3}$$

Then, for B an arbitrary $2^n \times 2^n$ matrix,

$$\psi(B) = \sum_{i_1,\ldots,i_L=0}^{n} \mathrm{Tr}(B v_{i_L} \cdots v_{i_1}) |i_1, \ldots, i_L\rangle \tag{4}$$

is a ground state of the model (MPS vector). In fact, they are all the ground states. E.g., one can pick B such that

$$\psi(B) = \sum_{x=1}^{L} \left(e^{i\theta_{i0}} \lambda_i\right)^x |0, \ldots, 0, i, 0, \ldots, 0\rangle$$

If we add the assumption that $\lambda_i \neq 1$, for $i = 1, \ldots, n$, we will have n_L particles having $\lambda_i < 1$ that bind to the left edge, and $n_R = n - n_L$ particles with $\lambda_i > 1$, which, when present, bind to the right edge. The bulk ground state is the vacuum state

$$\Omega = |0, \ldots, 0\rangle .$$

All other ground states differ from Ω only near the edges. We can show that the energy of the first excited state is bounded below by a positive constant, independently of the length of the chain. As at most one particle of each type can bind to the edge, any second particle of that type must be in a scattering state. The dispersion relation is

$$\epsilon_i(k) = 1 - \frac{2\lambda_i}{1+\lambda_i^2} \cos(k + \theta_{i0}) .$$

We conjecture that the *exact* gap of the infinite chain is

$$\gamma = \min \left\{ \frac{(1-\lambda_i)^2}{1+\lambda_i^2} \,\middle|\, i = 1, \ldots, n \right\} .$$

Automorphic equivalence classes of PVBS models

Two PVBS models with $\lambda_i \in (0,1) \cup (1,+\infty)$, $i = 1,\dots n$, belong to the same equivalence class if and only if they have the same n_L and n_R. $l_0 = l_1 = 2^{n_L}, r_0 = r_1 = 2^{n_R}$.

Recall that n_L is the number of i such that $\lambda_i \in (0,1)$ and n_R is the number of i such that $\lambda_i \in (1,+\infty)$. l_s and r_s are the dimensions of the ground state spaces of the left and right half-infinite chains.

(i) Since equivalent phases are related by an automorphism, a unique bulk ground state can only be mapped to another unique bulk state. Similarly, the ground state space dimensions of the half-infinite chains, 2^{n_L} and 2^{n_R}, are also preserved by an automorphism. Hence, if two PVBS models belong to the same phase, they must have equal n_L and n_R.

(ii) Conversely, if two PVBS models have the same values of n_L and n_R but each with their own sets of parameters $\{\lambda_i(s) \mid 1 \leq i \leq n_L + n_R\}$ and $\{\theta_{ij}(s) \mid 1 \leq i, j \leq n_L + n_R\}$, for $s = 0, 1$, first, perform a change of basis in spin space such that both sets of PVBS states are expressed in the same spin basis and such that $\lambda_i(s) < 1$ for $1 \leq i \leq n_L$ and $\lambda_i(s) > 1$ for $n_L + 1 \leq i \leq n_L + n_R$, for $s = 0$ and $s = 1$.
Let u be the unitary for this change of basis. Then, take a smooth curve of unitaries $u(s)$, $0 \leq s \leq 1$, with $u(0) = \mathbb{1}$ and $u(1) = u$, and let $U_{[a,b]}(s)$ be the $(b - a + 1)$-fold tensor product of $u(s)$. Then,

$$H_{[a,b]}(s) = U_{[a,b]}(s)^* H_{[a,b]}(0) U_{[a,b]}(s)$$

defines a smooth path of Hamiltonians with a constant gap.

Next, deform the parameters by simple linear interpolation:

$$\lambda_i(s) = (1-s)\lambda_i(0) + s\lambda_i(1) \qquad (5)$$

$$\theta_{ij}(s) = (1-s)\theta_{ij}(0) + s\theta_{ij}(1) \qquad (6)$$

This yields a smooth family of vectors $\phi_{ij}(s)$ and thereby a smooth family of nearest neighbor interactions $h(s)$. The gap remains open because $\lambda_i(s) \neq 1$ for all $i = 1, \ldots, n$ and $s \in [0,1]$. By our general result this implies the quasi-local automorphic equivalence of the two models.

If one uses the same type of interpolation to connect models with different values of n_L and n_R, the gap necessarily closes along the path and there is a quantum phase transition.

The AKLT model (Affleck-Kennedy-Lieb-Tasaki, 1987)

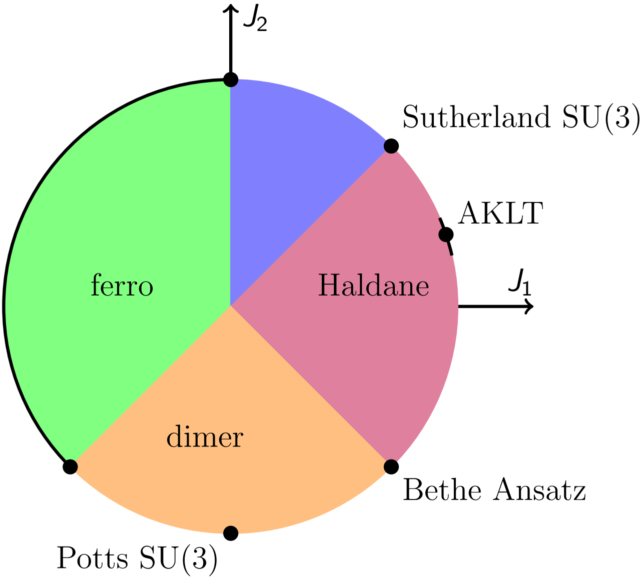
Antiferromagnetic spin-1 chain: $[1, L] \subset \mathbb{Z}$, $\mathcal{H}_x = \mathbb{C}^3$,

$$H_{[1,L]} = \sum_{x=1}^{L} \left(\frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \right) = \sum_{x=1}^{L} P^{(2)}_{x,x+1}$$

The ground state space of $H_{[1,L]}$ is 4-dimensional for all $L \geq 2$. In the limit of the infinite chain, the ground state is unique, has a finite correlation length, and there is a non-vanishing gap in the spectrum above the ground state (Haldane phase).

Theorem (Bachmann-N, CMP 2013, to appear)

There exists a curve of uniformly gapped Hamiltonians with nearest neighbor interaction $s \mapsto \Phi_s$ such that Φ_0 is the AKLT interaction and Φ_1 defines a PVBS model with $n_L = n_R = 1$ and a unique ground state of the infinite chain that is a product state.



$$H = \sum_x J_1 \mathbf{S}_x \cdot \mathbf{S}_{x+1} + J_2 (\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2$$

Symmetry protected phases in 1 dimension

For a given system with s-dependent G-symmetric interactions, we would like to find criteria to recognize that the system at s_0 is in a different gapped phase than at s_1, meaning that the gap above the ground state necessarily closes for at least one intermediate value of s.

This is the same problem as before but restricted to a class of models with a given symmetry group (and representation) G.

Our goal is to find invariants, i.e., computable and, in principle, observable quantities that can be different at s_0 and s_1, only if the model is in a different ground state phase.

Half-chains

Consider $\Gamma = [1, +\infty) \subset \mathbb{Z}$, and translation-invariant models defined by a nearest-neighbor interaction $h(s)$, $s \in [0, 1]$. Suppose

- $s \mapsto h(s)$ is differentiable;
- $h(s)$ commutes with a local symmetry G, i.e. $[h(s), \pi(g) \otimes \pi(g)] = 0, g \in G$, π a representation of G;
- there is a uniform lower bound $\gamma > 0$ for the spectral gap above the ground state of $\sum_{x=1}^{L-1} h_{x,x+1}(s)$, for all $L \geq 2$.

Let $\tau_g(A) = \bigotimes_{x \in \Gamma} \mathrm{ad}(\pi(g))A$, for all $g \in G$, the action of the symmetry on observables of the half-chain, and let σ_g^s denote the corresponding representation on the space spanned by the ground states: $\sigma_g^s(\omega) = \omega \circ \tau_g, \omega \in \mathcal{S}_s$.

Then, there exist quasi local automorphisms α_s such that

- $\alpha_s \circ \tau_g = \tau_g \circ \alpha_s$;
- $\mathcal{S}_s = \mathcal{S}_0 \circ \alpha_s$;
- $\sigma_g^s \cong \sigma_g^0$, for all $s \in [0, 1]$.

In other words:

Up to equivalence, the representation of G acting on the ground states of the half-chain is constant within a gapped ground state phase.

We have shown (Bachmann-N, arXiv:1307.0716, JSP, to appear) that for two interesting classes of models this invariant, the representation of G given by σ_g, can be observed from the ground state in the bulk, i.e. in the model defined on $\mathbb{Z}$.

The case $G = SU(2)$ and the Excess Spin

Models to keep in mind: antiferromagnetic chains in the Haldane phase and generalizations. Unique ground state with a spectral gap and an unbroken continuous symmetry.

Let S_x^i, $i = 1, 2, 3$, $x \in \mathbb{Z}$, denote the ith component of the spin at site x. Claim: one can define

$$\sum_{x=1}^{+\infty} S_x^i,$$

as s.a. operators on the GNS space of the ground state and they generate a representation of $SU(2)$ that is characteristic of the gapped ground state phase.

We can prove the existence of these excess spin operators for two classes of models:

1) models with a random loop representation;

2) models with a matrix product ground state (MPS).

Frustration-free chains with $SU(2)$ invariant MPS ground states

$$H = \sum_x h_{x,x+1}$$

Ground state is defined in terms of an isometry V, which intertwines two representations of $SU(2)$:

$$Vu_g = (U_g \otimes u_g)V, \quad g \in SU(2).$$

E.g., in the AKLT chain U_g is the spin-1 representation and u_g is the spin-$1/2$ representation of $SU(2)$, corresponding to the well-known spin $1/2$ degrees of freedom at the edges.
Let $k = \dim(u_g)$. The transfer operator is defined by

$$\mathbb{E}(B) = V^*(\mathbb{1} \otimes B)V, B \in M_k.$$

If ω is a G-invariant, pure, translation-invariant finitely correlated state generated by the intertwiner V, one can assume that 1 is the unique eigenvalue of maximal modulus of $\mathbb{E}$, and that it is simple (Fannes-N-Werner, JFA 1994).

Let $S = (S^1, S^2, S^3)$ be the vector of generators of U_g, and write $U_g = \exp(ig \cdot S)$. Define

$$S^+(L) = \sum_{x=1}^{L^2} f_L(x-1) S_x,$$

where $f_L : \mathbb{Z}^+ \to \mathbb{R}$ is given by

$f_L(mL+n) = 1 - m/L$, if $m, n \in [0, L-1]$, and $f_L(x) = 0$, if $x \geq L^2$.

Then, $U_g^+(L) = \exp(igS^+(L))$ is an observable and use the same notation for its representative on the GNS Hilbert space, $\mathcal{H}_\omega$, of ω.

Theorem

Let ω be as above. Then, the strong limit

$$U_g^+ = \lim_{L \to \infty} e^{ig \cdot S^+(L)}$$

exists and defines a strongly continuous unitary representation of G on $\mathcal{H}_\omega$.

The representation of U_g^+ is an invariant

Summary of the argument: $U_g^+|_{\pi_\omega(\mathcal{A}_{(-\infty,0]})\Omega_\omega} \cong (\bigoplus u_g)^\infty$.

(i) First consider the model on the half-infinite chain. The space of ground states transforms as u_g under the action of $SU(2)$. We call this the edge spin representation. We proved that, in general, along a curve of models with a non-vanishing gap, the edge representation is constant.

(ii) On the infinite chain, we showed that the excess spin representation is well-defined.

(iii) One can verify that on the subspace of the GNS Hilbert space of the infinite-chain ground state consisting of the ground state of the Hamiltonian of the half-infinite chain, acts as (an infinite number of copies of) u_g.

This is also shows that u_g is experimentally observable.

Concluding Remarks

- There are infinitely many inequivalent $SU(2)$ and translation invariant gapped ground state phases of integer spin chains.
- The frequently encountered statement that symmetry-protected gapped phases in one dimension are completely classified by the 2nd cohomology of the symmetry group G is somewhat misleading.
- We are close to a comprehensive picture of the gapped ground state phases in one dimension, but in two (and more) dimensions many questions remain open (work in progress on d-dimensional PVBS models with Bachmann, Hamza, and Young.)