Witnessing non-Markovianity of quantum evolution

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- to provide basic intro to some open problems in quantum open systems
- to provide new ideas in the field of non-Markovian evolution

Introduction – notation

n-level quantum system; $\quad n < \infty$

$$\mathcal{H} = \mathbb{C}^n \ ; \qquad \mathfrak{T}(\mathcal{H}) = \mathfrak{B}(\mathcal{H}) = M_n(\mathbb{C})$$

quantum states $\longrightarrow$ density operators in $M_n(\mathbb{C})$

$$\rho \geq 0 \ , \qquad \mathrm{Tr}\, \rho = 1$$

$\mathfrak{S}(\mathcal{H})$ – space of quantum states $\subset \mathfrak{T}(\mathcal{H})$

Positive maps

$$\Lambda : \mathfrak{T}(\mathcal{H}) \longrightarrow \mathfrak{T}(\mathcal{H})$$

$$X \geq 0 \quad \Longrightarrow \quad \Lambda(X) \geq 0$$

Λ – trace-preserving iff $\quad \mathrm{Tr}[\Lambda(X)] = \mathrm{Tr}\, X$

Λ maps states into states ; $\quad \Lambda(\mathfrak{S}(\mathcal{H})) \subset \mathfrak{S}(\mathcal{H})$

Positivity is too weak!

$$\Lambda : \mathfrak{T}(\mathcal{H}) \longrightarrow \mathfrak{T}(\mathcal{H})$$

$$\Lambda' : \mathfrak{T}(\mathcal{H}') \longrightarrow \mathfrak{T}(\mathcal{H}')$$

composite system $\longrightarrow$ $\mathcal{H} \otimes \mathcal{H}'$

$$\Lambda \otimes \Lambda' \; : \; \mathfrak{T}(\mathcal{H} \otimes \mathcal{H}') \longrightarrow \mathfrak{T}(\mathcal{H} \otimes \mathcal{H}')$$

$\Lambda \otimes \Lambda'$ needs NOT be a positive map!!!

Positivity is necessary but NOT sufficient for quantum physics

Positivity vs. complete positivity

A positive map

$$\Lambda : \mathfrak{T}(\mathcal{H}) \longrightarrow \mathfrak{T}(\mathcal{H})$$

is **k-positive** iff

$$\mathbb{1}_k \otimes \Lambda \;:\; M_k(\mathbb{C}) \otimes \mathfrak{T}(\mathcal{H}) \;\longrightarrow\; M_k(\mathbb{C}) \otimes \mathfrak{T}(\mathcal{H})$$

is positive. Λ is **completely positivity (CP)** iff it is k-positive for k=1,2,3,. . . .

$$\dim \mathcal{H} = n \quad \Longrightarrow \quad \text{CP} = n\text{-positive}$$

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Quantum channel

Λ – completely positive & trace-preserving (CPTP)

$$\Lambda(X) = \sum_\alpha K_\alpha X K_\alpha^\dagger$$

$$\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{I}$$

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$$\sum_{\alpha} K_{\alpha}^{\dagger} K_{\alpha} = \mathbb{I}$$

Quantum evolution - dynamical maps

$$\rho_t = \Lambda_t(\rho) \; ; \quad t \geq 0$$

$$\Lambda_t : \mathfrak{T}(\mathcal{H}) \longrightarrow \mathfrak{T}(\mathcal{H}) \; ; \quad \Lambda_0 = \mathbb{1}$$

Λ_t – CPTP

Basic example — Markovian semigroup

Markovian Master Equation

$$\dot{\Lambda}_t = L\Lambda_t \ ; \quad \Lambda_0 = \mathbb{1}$$

Gorini, Kossakowski, Sudarshan & Lindblad

$$L\rho \ = \ -i[H,\rho] + \sum_\alpha \left(V_\alpha \rho V_\alpha^\dagger - \frac{1}{2}\{V_\alpha^\dagger V_\alpha, \rho\} \right)$$

L – Kossakowski-Lindblad generator

$$\boxed{\Lambda_t = e^{Lt} \quad \Longrightarrow \quad \Lambda_{t+s} = \Lambda_t \, \Lambda_s}$$

Reduced dynamics

$$\mathcal{H} \otimes \mathcal{H}_R$$

$$\Lambda_t \rho := \mathrm{Tr}_R \left[e^{-iHt} (\rho \otimes \omega_R) e^{iHt} \right]$$

One obtains Markovian semigroup $\Lambda_t = e^{tL}$ only under suitable Born-Markov approximation

Genuine Λ_t is NOT of the form e^{tL} !

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Genuine Λ_t is NOT of the form e^{tL} !

How to go beyond Markovian semigroup ?

How to describe general quantum evolution?

$$\rho \quad \longrightarrow \quad \rho_t = \Lambda_t(\rho)$$

There are several approaches

- local in time master equation
- non-local master equation (Nakajima-Zwanzig approach)
- stochastic unraveling
- ...

Non-local approach

$$\Lambda_t \rho = \mathrm{Tr}_E [e^{-iHt} \rho \otimes \omega e^{iHt}]$$

Nakajima-Zwanzig projection technique

$$\frac{d}{dt}\Lambda_t = \int_0^t \mathcal{K}_{t-u} \Lambda_u du \ , \quad \Lambda_0 = \mathbb{1}$$

$\mathcal{K}_t$ – memory kernel

$\mathcal{K}_t = \delta(t) L \quad \longrightarrow \quad$ Markovian semigroup

Non-local approach

$$\frac{d}{dt}\Lambda_t = \int_0^t \mathcal{K}_{t-u}\Lambda_u du \ , \quad \Lambda_0 = \mathbb{1}$$

What is the structure of $\mathcal{K}_t$?

Solution Λ_t has to be CPTP !!!

Non-local approach

$$\frac{d}{dt}\Lambda_t = \int_0^t \mathcal{K}_{t-u}\Lambda_u du \ , \quad \Lambda_0 = \mathbb{1}$$

$$\widetilde{\Lambda}_s = \int_0^\infty e^{-st}\Lambda_t \, dt$$

$$s\widetilde{\Lambda}_s - \mathbb{1} = \widetilde{\mathcal{K}}_s\widetilde{\Lambda}_s$$

$$\widetilde{\Lambda}_s = [s\mathbb{1} - \widetilde{\mathcal{K}}_s]^{-1}$$

Λ_t – CPTP ; $\widetilde{\Lambda}_s$ – ???

CM functions

A function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is Completely Monotone iff

$$(-1)^n \frac{d^f}{dt^n}(t) \geq 0 \; ; \quad n = 0, 1, 2, \ldots$$

THEOREM [Bernstein] A function $f(t)$ is CM iff

$$f(s) = \int_0^t e^{-st} g(t) dt$$

and $g(t) \geq 0$.

Non-commutative version

DEFINITION: A family of super-operators F_t is CM iff

$$(-1)^n \frac{d^n}{dt^n} F_t \ \text{– CP} \ ; \quad n = 0, 1, 2, \ldots$$

THEOREM [non-commutative Bernstein] A family F_t is CM iff

$$F_s = \int_0^t e^{-st} G_t dt$$

and G_t is CP.

Non-local approach

$$\frac{d}{dt}\Lambda_t = \int_0^t \mathcal{K}_{t-u}\Lambda_u du \ , \quad \Lambda_0 = \mathbb{1}$$

$$\widetilde{\Lambda}_s = [s\mathbb{1} - \widetilde{\mathcal{K}}_s]^{-1}$$

Λ_t – CPTP ; $\widetilde{\Lambda}_s$ – CM

Difficult to control !!!

Local in time master equation

$$\frac{d}{dt}\Lambda_t = L_t \Lambda_t \ , \quad \Lambda_0 = \mathbb{1}$$

$$\Lambda_t = \mathcal{T} \exp\left(\int_0^t L_u du\right) =$$

$$= \mathbb{1} + \int_0^t dt_1\, L_{t_1} + \int_0^t dt_1 \int_0^{t_1} dt_2\, L_{t_1} L_{t_2} + \dots$$

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Basic question

What are condition for L_t such that the solution to

$$\frac{d}{dt}\Lambda_t = L_t \Lambda_t \ , \quad \Lambda_0 = \mathbb{1}$$

$$\Lambda_t = \mathrm{T}\exp\left(\int_0^t L_u du\right)$$

is legitimate — CPTP?

Conditions for L_t are NOT known

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What are condition for L_t such that the solution to

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Conditions for L_t are NOT known

Special classes

1. Markovian semigroup (K-L generator)
2. CP-divisible maps
3. Commutative dynamics

Divisible maps (CP-divisible)

Λ_t – dynamical map

Λ_t is divisible iff

$$\boxed{\Lambda_t = V_{t,s}\Lambda_s}$$

$$t \geq s \geq 0$$

$V_{t,s}$ completely positive maps for all $t \geq s$

Divisible maps (CP-divisible)

THEOREM: a map Λ_t is CP-divisible iff the corresponding time-local generator L_t

$$\frac{d}{dt}\Lambda_t = L_t\Lambda_t$$

has Kossakowski-Lindblad form for all $t \geq 0$:

$$L_t\rho \;=\; -i[H(t),\rho] + \sum_\alpha \left(V_\alpha(t)\rho V_\alpha^\dagger(t) - \frac{1}{2}\{V_\alpha^\dagger(t)V_\alpha(t),\rho\} \right)$$

$$V_{t,s}\cdot V_{s,u} = V_{t,u}\;;\quad t\geq s\geq u$$

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P-divisible maps

Λ_t – dynamical map

Λ_t is P-divisible iff

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$$t \geq s \geq 0$$

$V_{t,s}$ positive maps for all $t \geq s$

CP-divisible $\Longrightarrow$ P-divisible

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CP-divisible $\Longrightarrow$ P-divisible

Commutative dynamics

$$[L_t, L_u] = 0$$

$$\Lambda_t = \mathcal{T} \exp\left(\int_0^t L_u du\right) = \exp\left(\int_0^t L_u du\right)$$

L_t defines a legitimate generator iff

$$\int_0^t L_u du \quad \text{has K-L form for all } t \geq 0$$

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Example: pure decoherence

$$L_t(\rho) = \gamma(t)(\sigma_z \rho \sigma_z - \rho)$$

$$\Gamma(t) = \int_0^t \gamma(u) du$$

$$\rho_t = \begin{pmatrix} \rho_{11} & e^{-\Gamma(t)} \rho_{12} \\ e^{-\Gamma(t)} \rho_{21} & \rho_{22} \end{pmatrix}$$

- Λ_t is CPTP iff $\Gamma(t) \geq 0$
- Λ_t is CP-divisible iff $\gamma(t) \geq 0$

$\gamma(t) \ngeq 0 \quad \longleftrightarrow \quad$ the essence of non-Markovianity

Markovian vs. non-Markovian

- Markovianity is defined for classical stochastic processe!
- Markovianity = semigroup dynamics
- Markovianity = CP-divisibility (Rivas, Huelga, Plenio) $\longrightarrow$ non-Markovianity measure
- Markovianity = negative information flow (Breuer, Laine, Piilo) $\longrightarrow$ non-Markovianity measure
- Geometrical characterization of non-Markovianity (Lorenzo, Plastina, Paternostro) $\longrightarrow$ non-Markovianity measure
- non-Markovianity measure via mutual information (Luo)
- non-Markovianity measure via channel capacity (Bylicka, DC, Maniscalco)
- ...

Breuer-Laine-Piilo (BLP) condition

Evolution is Markovian if

$$\sigma(\rho_1, \rho_2; t) := \frac{d}{dt} ||\Lambda_t(\rho_1 - \rho_2)||_1 \leq 0$$

for all pairs ρ_1 and ρ_2.

$$\text{CP-divisibility} \implies \sigma(\rho_1, \rho_2; t) \leq 0$$

The converse needs NOT be true!
DC, A. Kossakowski and A. Rivas, PRA 2011.

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Example: Pure dephasing

$$L_t(\rho) = \gamma(t)(\sigma_z \rho \sigma_z - \rho)$$

$$\text{CP-divisibility} \iff \sigma(\rho_1, \rho_2; t) \leq 0 \iff \gamma(t) \geq 0$$

The evolution is well defined iff

$$\Gamma(t) = \int_0^t \gamma(u) du \geq 0$$

$$\Phi : \mathfrak{T}(\mathcal{H}) \longrightarrow \mathfrak{T}(\mathcal{H})$$

THEOREM: Φ – positive and trace-reserving

$$\Longrightarrow \quad ||\Phi(X)||_1 \leq ||X||_1$$

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Λ_t – dynamical map

$$||[\mathbb{1} \otimes \Lambda_t](X)||_1 \leq ||X||_1 \quad ; \quad X \in \mathfrak{T}(\mathcal{H} \otimes \mathcal{H})$$

THEOREM: [DC, A. Kossakowski, A. Rivas, PRA 2011]
Let Λ_t be invertible, then Λ_t is divisible iff

$$\frac{d}{dt} ||[\mathbb{1} \otimes \Lambda_t](X)||_1 \leq 0 \quad ; \quad X \in \mathfrak{T}(\mathcal{H} \otimes \mathcal{H})$$

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Non-Markovianity witness

Λ_t – dynamical map

DEFINITION: $X^* = X$ is a non-Markovianity witness for Λ_t if

$$\frac{d}{dt}||[\mathbb{1}\otimes\Lambda_t](X)||_1 > 0$$

for some $t > 0$.

COROLLARY: Λ_t is non-Markovian iff there is a non-Markovianity witness for Λ_t.

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Non-Markovianity witness

Non-Markovianity witness is an analog of the entanglement witness $W = W^* \in \mathfrak{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is an entanglement witness iff

- $W \ngeq 0$
- $\langle \psi_A \otimes \phi_B | W | \psi_A \otimes \phi_B \rangle \geq 0$

A state ρ is ENTANGLED iff there is an entanglement witness W such that

$$\mathrm{Tr}(W\rho) < 0$$

CP-divisibility

Diamond norm

$$||\Phi||_\diamond := \sup_{||X||_1} ||(\mathbb{1} \otimes \Phi)X||_1$$

THEOREM: Let Λ_t be an invertible dynamical map. Λ_t is CP-divisble iff

$$V_{t,s} = \Lambda_t \circ \Lambda_s^{-1}$$

satisfies

$$||V_{t,s}||_\diamond = 1 \ , \quad t \geq s$$

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k-divisibility

DEFINITION: Λ_t is k-divisible iff

$$\Lambda_t = V_{t,s}\Lambda_s$$

and $V_{t,s}$ is k-positive for $t \geq s$.

n-divisibility $\Longleftrightarrow$ CP-divisibility

1-divisibility $\Longleftrightarrow$ P-divisibility

analog of the Schmidt number of the density operator

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analog of the Schmidt number of the density operator

Example 1: pure decoherence

$$L_t(\rho) = \gamma(t)(\sigma_z \rho \sigma_z - \rho)$$

CP-divisibility $\Longleftrightarrow$ P-divisibility $\Longleftrightarrow$ $\gamma(t) \geq 0$

$\gamma(t) \geq 0$ – time-dependent decoherence rate

$\gamma(t) < 0$ – time-dependente "recoherence" rate

The evolution is well defined iff

$$\Gamma(t) = \int_0^t \gamma(u)du \geq 0 \quad \longrightarrow \quad \rho_t = \begin{pmatrix} \rho_{11} & e^{-\Gamma(t)}\rho_{12} \\ e^{-\Gamma(t)}\rho_{21} & \rho_{22} \end{pmatrix}$$

Example 1

non-Markovianity witness

$$X = \frac{1}{2}\sigma_x \otimes \sigma_x$$

$$\frac{d}{dt}||(\mathbb{1} \otimes \Lambda_t)X||_1 = -\gamma(t)e^{-\Gamma(t)}$$

$$\frac{d}{dt}||(\mathbb{1} \otimes \Lambda_t)X||_1 > 0 \quad \Longleftrightarrow \quad \gamma(t) < 0$$

Example 2: random unitary dynamics

$$\Lambda_t \rho = \sum_{\alpha=0}^{3} p_\alpha(t) \sigma_\alpha \rho \sigma_\alpha$$

$$p_\alpha(t) \geq 0 \ ; \quad p_0(0) = 1 \ ; \quad \sum_{\alpha=0}^{3} p_\alpha(t) = 1$$

$$\dot{\Lambda}_t = L_t \Lambda_t$$

$$L_t \rho = \sum_{\alpha=1}^{3} \gamma_k(t) [\sigma_k \rho \sigma_k - \rho]$$

$\gamma_k(t) =$ function of $p_\alpha(t)$

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$$L_t \rho = \sum_{\alpha=1}^{3} \gamma_k(t) [\sigma_k \rho \sigma_k - \rho]$$

$$\gamma_k(t) = \text{function of } p_\alpha(t)$$

$$\Lambda_t \rho = \sum_{\alpha=0}^{3} p_\alpha(t) \sigma_\alpha \rho \sigma_\alpha \longleftrightarrow L_t \rho = \sum_{\alpha=1}^{3} \gamma_k(t) [\sigma_k \rho \sigma_k - \rho]$$

$\lambda_1(t) = e^{-2[\Gamma_2(t)+\Gamma_3(t)]}$ & cyclic perm

$$\begin{aligned}
p_0(t) &= \frac{1}{4}\left[1 + \lambda_1(t) + \lambda_2(t) + \lambda_3(t)\right] \\
p_1(t) &= \frac{1}{4}\left[1 + \lambda_1(t) - \lambda_2(t) - \lambda_3(t)\right] \\
p_2(t) &= \frac{1}{4}\left[1 - \lambda_1(t) + \lambda_2(t) - \lambda_3(t)\right] \\
p_3(t) &= \frac{1}{4}\left[1 - \lambda_1(t) - \lambda_2(t) + \lambda_3(t)\right]
\end{aligned}$$

$$L_t\rho = \sum_{\alpha=1}^{3} \gamma_k(t)[\sigma_k\rho\sigma_k - \rho]$$

The random unitary dynamics is CP-divisible iff

$$\gamma_1(t) \geq 0 \ , \quad \gamma_2(t) \geq 0 \ , \quad \gamma_3(t) \geq 0 \ .$$

The random unitary dynamics is P-divisible iff

$$\gamma_1(t) + \gamma_2(t) \geq 0 \ , \quad \gamma_1(t) + \gamma_3(t) \geq 0 \ , \quad \gamma_2(t) + \gamma_3(t) \geq 0$$

The random unitary dynamics is CP iff

$$\Gamma_1(t) \geq 0 \ , \quad \Gamma_2(t) \geq 0 \ , \quad \Gamma_3(t) \geq 0 \ .$$

[DC, F. Wudarski, PLA 2013]

$$\lambda_1(t) = e^{-2[\Gamma_2(t)+\Gamma_3(t)]} \quad \& \quad \text{cyclic perm}$$

$$\begin{aligned}
p_0(t) &= \frac{1}{4}\Big[1 + \lambda_1(t) + \lambda_2(t) + \lambda_3(t)\Big] \\
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\end{aligned}$$

$$\gamma_1(t) + \gamma_2(t) \geq 0 \ , \quad \gamma_1(t) + \gamma_3(t) \geq 0 \ , \quad \gamma_2(t) + \gamma_3(t) \geq 0$$

$$\Gamma_1(t) + \Gamma_2(t) \geq 0 \ , \quad \Gamma_1(t) + \Gamma_3(t) \geq 0 \ , \quad \Gamma_2(t) + \Gamma_3(t) \geq 0 \ \longleftrightarrow \ ???$$

Bloch equations

$$\rho_t = \frac{1}{2}\left(\mathbb{I} + \sum_k x_k(t)\sigma_k\right)$$

$$\dot{\rho}_t = \sum_k \gamma_k[\sigma_k\rho_t\sigma_k - \rho_t]$$

$$\dot{x}_k(t) \quad = \quad -\frac{1}{T_k(t)}\,x_k(t)\ ; \quad k = 1,2,3$$

relaxation time $\longrightarrow$ $T_1 = \dfrac{1}{2(\gamma_2 + \gamma_3)}$ & cyclic perm

CP-divisibility

decoherence rates $\gamma_k(t) \geq 0$

P-divisibility

relaxation times $T_k(t) \geq 0$

CP-divisibility

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Example 3

$$L_t\rho = \gamma_+(t)\left([\sigma_+, \rho\sigma_-] + [\sigma_+\rho, \sigma_-]\right) + \gamma_-(t)\left([\sigma_-, \rho\sigma_+] + [\sigma_-\rho, \sigma_+]\right) ,$$

$$\sigma_+ = |2\rangle\langle 1| \ ; \qquad \sigma_- = |1\rangle\langle 2|$$

CP-divisibility $\longleftrightarrow$ $\gamma_-(t) \geq 0 \ \ \& \ \ \gamma_+(t) \geq 0$

P-divisibility $\longleftrightarrow$ $\gamma_-(t) + \gamma_+(t) \geq 0$

CPT map $\longleftrightarrow$ $0 \leq \int_0^t \gamma_+(s) e^{\Gamma(s)} ds \leq e^{\Gamma(t)} - 1$

$$\Gamma(t) = \Gamma_-(t) + \Gamma_+(t)$$

Example 3

$$L_t\rho = \gamma_+(t)([\sigma_+, \rho\sigma_-] + [\sigma_+\rho, \sigma_-]) + \gamma_-(t)([\sigma_-, \rho\sigma_+] + [\sigma_-\rho, \sigma_+]) \ ,$$

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$$\text{CP-divisibility} \ \longleftrightarrow \ \gamma_-(t) \geq 0 \ \ \& \ \ \gamma_+(t) \geq 0$$

$$\text{P-divisibility} \ \longleftrightarrow \ \gamma_-(t) + \gamma_+(t) \geq 0$$

$$\text{CPT map} \ \longleftrightarrow \ 0 \leq \int_0^t \gamma_+(s) e^{\Gamma(s)} ds \leq e^{\Gamma(t)} - 1$$

$$\Gamma(t) = \Gamma_-(t) + \Gamma_+(t)$$

Examples 1 and 2 are commutative

$$L_t \rho = \gamma(t)[\sigma_z \rho \sigma_z - \rho]$$

$$L_t \rho = \sum_{\alpha=1}^{3} \gamma_k(t)[\sigma_k \rho \sigma_k - \rho]$$

example 3 is NOT

$$L_t \rho = \gamma_+(t)([\sigma_+, \rho\sigma_-] + [\sigma_+\rho, \sigma_-]) + \gamma_-(t)([\sigma_-, \rho\sigma_+] + [\sigma_-\rho, \sigma_+])$$

Lie algebraic methods

$$L_t = a_1(t)L_1 + a_2(t)L_2$$

$$L_1\rho = [\sigma_+, \rho\sigma_-] + [\sigma_+\rho, \sigma_-]$$

$$L_2\rho = [\sigma_-, \rho\sigma_+] + [\sigma_-\rho, \sigma_+]$$

$$[L_1, L_2] = L_1 - L_2$$

L_1 and L_2 close a Lie algebra.

Lie algebraic methods

$$L_t = a_1(t)L_1 + a_2(t)L_2$$

$$\Lambda_t = \mathrm{T}\exp\left(\int_0^t L_u du\right)$$

$$\Lambda_t = \exp\left(\int_0^t X_u du\right) = \exp\left(B_1(t)L_1 + B_2(t)L_2\right)$$

$$X_t = b_1(t)L_1 + b_2(t)L_2$$

$$\Lambda_t \ - \text{CPTP} \iff B_1(t) \geq 0\ ,\ B_2(t) \geq 0$$

$$B_k(t) = \int_0^t b_k(u)du$$

Lie algebraic methods

$$L_t = a_1(t)L_1 + a_2(t)L_2$$

$$\Lambda_t = \mathrm{T}\exp\left(\int_0^t L_u du\right)$$

$$\Lambda_t = \exp\left(\int_0^t X_u du\right) = \exp\left(B_1(t)L_1 + B_2(t)L_2\right)$$

$$X_t = b_1(t)L_1 + b_2(t)L_2$$

Λ_t – CPTP $\iff B_1(t) \geq 0\ ,\ B_2(t) \geq 0$

$$B_k(t) = \int_0^t b_k(u)du$$

Lie algebraic methods

$$\Lambda_t = \mathrm{T}\exp\left(\int_0^t [a_1(u)L_1 + a_2(u)L_2]du\right)$$

$$\Lambda_t = \exp\left(\int_0^t [b_1(u)L_1 + b_2(u)L_2]du\right) = \exp\left[B_1(t)L_1 + B_2(t)L_2\right]$$

$$A_k(t) = \int_0^t a_k(u)du \ ; \quad B_k(t) = \int_0^t b_k(u)du$$

$$a_k(t) \quad \longleftrightarrow \quad b_k(t)$$

$$\Lambda_t = \mathrm{T}\exp\left(\int_0^t [a_1(u)L_1 + a_2(u)L_2]du\right)$$

$$\Lambda_t = \exp\left(\int_0^t [b_1(u)L_1 + b_2(u)L_2]du\right) = \exp\left[B_1(u)L_1 + B_2(u)L_2\right]$$

$$\begin{aligned} b_1(t) &= a_1(t) - f(t) \\ b_2(t) &= a_2(t) + f(t) \end{aligned}$$

$$f = e^{-A}\frac{W}{A} \; ; \quad W = a_1 A_2 - a_2 A_1 \; ; \quad A = A_1 + A_2$$

$$A_1 + A_2 = B_1 + B_2 \implies A_1 + A_2 \geq 0$$

$$\Lambda_t = \mathrm{T}\exp\left(\int_0^t [a_1(u)L_1 + a_2(u)L_2]du\right)$$

$$\Lambda_t = \exp\left(\int_0^t [b_1(u)L_1 + b_2(u)L_2]du\right) = \exp\left[B_1(t)L_1 + B_2(t)L_2\right]$$

$$a_1 = b_1 + f \ ; \quad a_2 = b_2 - f$$

$$\Lambda_t = \mathrm{T}\exp\left(\int_0^t [b_1(u)L_1 + b_2(u)L_2 + f(u)[L_1, L_2]]du\right)$$

$$\Lambda_t = \exp\left(\int_0^t [b_1(u)L_1 + b_2(u)L_2]du\right)$$

Conclusions

- non-local approach needs CM super-operator functions
- local approach – open problem
- witnessing non-Markovianity (non-divisibility)
- k-divisibility — various degrees of non-Markovianity

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